

Rational Equivariant Spectra

David James Barnes

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Department of Pure Mathematics

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Introduction

0.1 Rational Equivariant Cohomology Theories

Cohomology theories provide information about the abstract nature of topological spaces and allow us to distinguish between and work with these spaces. To each space X , a cohomology theory E^* gives a graded group $E^*(X)$. For a map of spaces $f: X \rightarrow Y$ we have a map of graded groups $E^*(f): E^*(Y) \rightarrow E^*(X)$ and this structure satisfies many useful axioms. Since spaces with a group action abound in all areas of mathematics, an understanding of how these spaces differ and behave is of particular importance. A G -equivariant cohomology theory is a special kind of cohomology theory that is designed to be used on spaces with a G -action.

Particular examples include: equivariant K -theory [Seg68] (which is constructed in the same way as K -theory but using G -vector bundles) and equivariant cobordism [tD72]. The Borel construction takes any cohomology theory E^* and makes a G -equivariant cohomology theory from it: $E_G^*(X) = E^*(EG_+ \wedge_G X)$ (EG_+ is the universal free G -space with an additional G -fixed point adjoined). A rational G -equivariant cohomology theory satisfies the additional condition that each group $E^*(X)$ is a rational vector space. This extra assumption simplifies the area so that it becomes amenable to study and calculations (spectral sequences) become much easier once one works over a field.

The result below, which only applies to finite groups G , follows from [GM95, Appendix A]. Let $g\mathbb{Q}W_G H\text{-mod}$ denote the category of graded left modules over the rational group ring of $W_G H$, the Weyl group of H in G .

Theorem 1 The category of rational G -equivariant cohomology theories is equivalent to the category $\prod_{(H) \leq G} g\mathbb{Q}W_G H\text{-mod}$, where the product runs over the collection of conjugacy classes of subgroups of G .

This shows that one can classify the collection of rational equivariant cohomology theories in terms of an algebraic model that is easy to understand. Such classifications are performed in [Gre99] for the circle group and [Gre98b] for $O(2)$ via spectral sequence calculations. These calculations become almost impossible when the rank of the group is greater than 1 (that is, the group contains a torus). Thus it would be advantageous to have an alternative method to classify rational equivariant cohomology theories.

0.2 Classifying Cohomology Theories with Model Categories

To study the category of cohomology theories as a whole, one works with their representing objects – spectra. The basic idea is that an element of $E^*(X)$, the E^* -cohomology of the space X , is a homotopy class of maps from $\Sigma^\infty X$ to an equivariant spectrum E . We say that E represents E^* and to understand the cohomology theory E^* we study the spectrum E . Thus we want to understand the homotopy category of equivariant spectra. The categorical foundations of equivariant spectra have developed substantially in the past twenty-five years. The encyclopedic [LMSM86] constructed a good category of equivariant spectra and put the subject area on firm footing. Unfortunately the smash product constructed there is only a monoidal product up to homotopy. Models for spectra with point-set level associative smash products are constructed in [EKMM97], [HSS00] and [MMSS01]. This construction was soon adapted to an equivariant setting in [May96] and [MM02]. These newer categories come with model structures that can help us to study their homotopy categories.

Very roughly, a model category has a collection of weak equivalences that one formally makes into isomorphisms to create the homotopy category. So, one inverts the collection of weak homotopy equivalences in the category of spaces to obtain the category of CW -complexes with morphisms given by homotopy classes of maps. In particular, the homotopy category of a model category of G -spectra (there are several choices for such a model category, all giving the same homotopy category) is the category of G -equivariant cohomology theories. A Quillen equivalence between model categories is an adjoint pair of functors such that these functors create an equivalence of the homotopy categories. The precise definition of model categories and Quillen equivalences give specific criteria to check to see that an adjoint pair is a Quillen equivalence. It is generally accepted that the notion of Quillen equivalence is the correct way to say that two categories have the same homotopy theory.

Since a Quillen equivalence is defined in terms of an adjoint pair, one often has a Quillen equivalence $A \rightleftarrows B$ and a Quillen equivalence $B \rightleftarrows C$ (with left adjoints on top), but one cannot combine these adjoint pairs to get an adjunction between A and C . Instead one says that A and C are Quillen equivalent by a zig-zag of Quillen equivalences. In this example one only has two Quillen equivalences, but in general one can have any finite number of Quillen equivalences in a zig-zag.

Model categories can be used to encode more information into the homotopy category. One may be interested in model categories with a monoidal product, such as the tensor product of modules over a commutative ring or the smash product of spaces or spectra. When this product is compatible with the model category we have a monoidal model category. A monoidal Quillen equivalence (roughly) is a Quillen equivalence which identifies the monoidal product in the homotopy categories. Hence results about the monoidal product structure in one homotopy category apply equally well to the other in a monoidal Quillen equivalence.

Our algebraic model for G -spectra, $dg\mathcal{A}(G)$, is now expected to have the structure of a model category. Thus, one can now ask for a zig-zag of Quillen equivalences between a

model category of G -equivariant spectra and a model category $dg\mathcal{A}(G)$. This Quillen equivalence tells us that if one is interested in homotopy level information one only needs to work with $dg\mathcal{A}(G)$. If we can make this zig-zag from a series of monoidal Quillen functors then we can model the smash product of spectra by a tensor product operation in $dg\mathcal{A}(G)$. This allows us to model more complicated structures in spectra by analogous structures in $dg\mathcal{A}(G)$, such as understanding the collection of modules over a ring spectrum by considering modules over a ring object in $dg\mathcal{A}(G)$.

0.3 Existing Work

The paper [Shi02] proves that the category of rational $SO(2)$ -spectra and $dg\mathcal{A}(SO(2))$ (as constructed in [Gre99]) are Quillen equivalent using information about Massey products for differential graded rings with many objects. This paper does not consider monoidal structures and relies on the ease of calculation for the circle group.

The classification of rational G -spectra for finite G is extended to a Quillen equivalence in [SS03b, Example 5.1.2]. This paper provides a first blueprint for our new method of classifying rational G -cohomology theories. Starting from $G\mathcal{S}_{\mathbb{Q}}$, a category of rational G -spectra, one considers \mathcal{G}_{top} , a collection of generators (for the homotopy category). By using the good properties of $G\mathcal{S}_{\mathbb{Q}}$ we can construct $\mathcal{E}_{top}(\sigma_1, \sigma_2)$, a symmetric spectrum of functions for each pair of pair of generators (σ_1, σ_2) . This collection has a composition rule, $\mathcal{E}_{top}(\sigma_2, \sigma_3) \wedge \mathcal{E}_{top}(\sigma_1, \sigma_2) \rightarrow \mathcal{E}_{top}(\sigma_1, \sigma_3)$.

Thus we have created an enriched category which we call \mathcal{E}_{top} , it has object set \mathcal{G}_{top} and the subscript top indicates that this category is of topological origin. We can consider the category of enriched functors from \mathcal{E}_{top} to symmetric spectra, we call such a functor a right \mathcal{E}_{top} -module and denote the category of such by $\text{mod-}\mathcal{E}_{top}$. If M is one of these enriched functors then for each pair σ_1, σ_2 in \mathcal{G}_{top} we have symmetric spectra $M(\sigma_1)$ and $M(\sigma_2)$ with an action map $M(\sigma_2) \wedge \mathcal{E}_{top}(\sigma_1, \sigma_2) \rightarrow M(\sigma_1)$.

This category of modules is referred to as the collection of ‘topological Mackey functors’ in [SS03b]. The categories $G\mathcal{S}_{\mathbb{Q}}$ and $\text{mod-}\mathcal{E}_{top}$ are Quillen equivalent by [SS03b, Theorem 3.3.3]. Since G is finite and we are working rationally, the homotopy groups of $\mathcal{E}_{top}(\sigma_1, \sigma_2)$ are concentrated in degree zero where they take value $\underline{A}(\sigma_1, \sigma_2)$, a \mathbb{Q} -module. Hence $\mathcal{E}_{top}(\sigma_1, \sigma_2)$ it is weakly equivalent to an Eilenberg-Mac Lane spectrum $H\underline{A}(\sigma_1, \sigma_2)$.

From the collection of spectra $H\underline{A}(\sigma_1, \sigma_2)$, we construct a category $H\underline{A}$, which is enriched over symmetric spectra and one replaces $\text{mod-}\mathcal{E}_{top}$ by the Quillen equivalent category $\text{mod-}H\underline{A}$. The collection $\underline{A}(\sigma_1, \sigma_2)$ for $\sigma_1, \sigma_2 \in \mathcal{G}_{top}$ can be thought of as a category enriched over $dg\mathbb{Q}$ -mod and thus we have a model category $\text{mod-}\underline{A}$. The category of rational Mackey functors is the collection of additive functors $\pi_0\mathcal{E}_{top} \rightarrow \mathbb{Q}$. Since \underline{A} is equal to $\pi_0\mathcal{E}_{top}$, $\text{mod-}\underline{A}$ is the category of rational differential graded Mackey functors. There is a zig-zag of Quillen equivalences between $\text{mod-}H\underline{A}$ and $\text{mod-}\underline{A}$. Thus rational G -spectra have been classified in terms of an algebraic category. Since these categories are rational the homotopy category of $\text{mod-}\underline{A}$ is equivalent to the category of graded rational Mackey functors, which is equivalent to $\prod_{(H) \leq G} g\mathbb{Q}W_G H\text{-mod}$. Thus

this paper recovers the results of [GM95, Appendix A]. This classification does not consider monoidal structures and requires the assumption that the homotopy groups of $\mathcal{E}_{top}(\sigma_1, \sigma_2)$ are concentrated in degree zero.

The preprint [GS] is intended to be a combination of [SS03b, Example 5.1.2] and the paper [Shi07b], it will classify rational torus equivariant spectra in terms of an algebraic category. Perhaps more importantly it provides a basis for the classification process in general as it doesn't have the strict requirements of the first two methods. This preprint is currently under substantial review and expansion. We have worked from an enhanced version, but our references agree with the publicly available version. As we will comment upon later (Section 0.5), the method of [GS] is much more compatible with the monoidal structures. We now outline the method of this paper.

Once again one begins with $G\mathcal{S}_{\mathbb{Q}}$ and uses the Quillen equivalence between $G\mathcal{S}_{\mathbb{Q}}$ and $\text{mod-}\mathcal{E}_{top}$. One can now apply the results of [Shi07b] to construct a category \mathcal{E}_t from \mathcal{E}_{top} . This new category will be enriched over rational chain complexes with its set of objects given by \mathcal{G}_{top} . The t indicates that we have come from the topological side but are now working in an algebraic setting. We can consider enriched functors from \mathcal{E}_t to rational chain complexes, this category will be denoted $\text{mod-}\mathcal{E}_t$.

Now we begin our work from the other end, assuming that we have a suitable candidate, $dg\mathcal{A}(G)$, for the algebraic model. We choose a generating set \mathcal{G}_a for $dg\mathcal{A}(G)$, we require that this set has a specified isomorphism to \mathcal{G}_{top} . Analogously to the topological setting, the set \mathcal{G}_a is the object set for a category \mathcal{E}_a , which is enriched over rational chain complexes. We can then replace $dg\mathcal{A}(G)$ by $\text{mod-}\mathcal{E}_a$. The a indicates we have come from the algebraic model. The notation \mathcal{E}_{top} , \mathcal{E}_t and \mathcal{E}_a is taken from [GS], which uses top , t and a to indicate whether a particular object is topological, algebraic but from the topological side or purely algebraic.

So far this process has been formal, now one must use some specific information about \mathcal{E}_t and \mathcal{E}_a to achieve a comparison between them. The comparison we will use is the notion of a quasi-isomorphism of categories enriched over rational chain complexes. Given two such categories \mathcal{C} and \mathcal{D} with isomorphic object sets, an enriched functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a quasi-isomorphism if each $F(\sigma_1, \sigma_2): \mathcal{C}(\sigma_1, \sigma_2) \rightarrow \mathcal{D}(F\sigma_1, F\sigma_2)$, is a homology isomorphism ($F(\sigma_1, \sigma_2)$ is a map in the category of rational chain complexes).

One shows by calculation that $\mathcal{E}_t(\sigma_1, \sigma_2)$ and $\mathcal{E}_a(\sigma_1, \sigma_2)$ (which are rational chain complexes) have the same homology for each pair (σ_1, σ_2) . Then one proves that this homology is intrinsically formal, that is, any two such enriched categories with this homology must be quasi-isomorphic. Now one returns to formal considerations and shows that since \mathcal{E}_t and \mathcal{E}_a are quasi-isomorphic, there is a zig-zag of Quillen equivalences between $\text{mod-}\mathcal{E}_t$ and $\text{mod-}\mathcal{E}_a$. Putting all of this together gives the desired result, a zig-zag of Quillen equivalences between G -spectra and our algebraic model.

0.4 Contents of this Thesis

The Splitting

We will study the case of finite groups and the case $G = O(2)$. A new ingredient to the method outlined above is the notion of splitting the category of rational G -spectra $G\mathcal{S}_{\mathbb{Q}}$. The language of Bousfield localisations is used here, it is a method of altering the homotopy category of a model category. Given a model category M , the homotopy category $\mathrm{Ho} M$ is formed by inverting the weak equivalences. Hence if we change the weak equivalences without changing the objects and morphisms of M we obtain a new model category M' and a new homotopy category $\mathrm{Ho} M'$. A spectrum E defines a homology theory E_* , from this we have the notion of E -equivalences, those maps f such that E_*f is an isomorphism. The Bousfield localisations that we will use are called E -localisations, where the new weak equivalences are the E -equivalences. We write the E -localisation of $G\mathcal{S}_{\mathbb{Q}}$ as $L_E G\mathcal{S}_{\mathbb{Q}}$.

Theorem 2 (3.2.4) Let $\{E_i\}_{i \in I}$ be a finite collection G -spectra. If $E_i \wedge E_j$ is rationally acyclic for $i \neq j$ and $\bigvee_{i \in I} E_i$ is rationally equivalent to \mathbb{S} then we have a monoidal Quillen equivalence

$$\Delta : G\mathcal{S}_{\mathbb{Q}} \rightleftarrows \prod_{i \in I} L_{E_i} G\mathcal{S}_{\mathbb{Q}} : \prod.$$

Now we assume that for each i we have a model category $dg\mathcal{A}(G)_i$ which is supposed to model $L_{E_i} G\mathcal{S}_{\mathbb{Q}}$. Fix some E_i and apply the method outlined above to get to the stage where we must compare $\mathcal{E}_t(i)$ and $\mathcal{E}_a(i)$. Here is where we see the advantage of the splitting, it simplifies the categories $\mathcal{E}_t(i)$ and $\mathcal{E}_a(i)$ and thus makes it easier to prove they are equivalent. We describe how this works in the finite case and the $O(2)$ case.

The Finite Case.

We apply our splitting theorem and the method of [GS] to reprove the result of [SS03b, Example 5.1.2]. This will be a good introduction to the general method and once the monoidality issue has been resolved it will be a monoidal classification. It should also be helpful in seeing how to proceed in a classification of the category of dihedral $O(2)$ -spectra, which we shall define later. Recall that $A(G) \otimes \mathbb{Q} \cong \prod_{(H)} \mathbb{Q}$ by tom-Dieck's isomorphism, where the product runs over the set of conjugacy classes of subgroups of G . Thus for each conjugacy class of subgroups (H) we have an idempotent e_H and we set $E_H = e_H \mathbb{S}$. Our splitting theorem then states that the category of G -spectra is monoidally equivalent to $\prod_{(H)} L_{E_H} G\mathcal{S}_{\mathbb{Q}}$. So we work through the method outlined above for each $L_{E_H} G\mathcal{S}_{\mathbb{Q}}$ individually. In fact, we can use the same argument for each conjugacy class (H) . Each $L_{E_H} G\mathcal{S}_{\mathbb{Q}}$ is generated by a single element, $G/H_+ \wedge e_H \mathbb{S}$. Thus the set of objects for $\mathcal{E}_{top}(H)$ will be the collection of smash products of $G/H_+ \wedge e_H \mathbb{S}$.

Following the previous work on the finite case an obvious candidate for the algebraic model of the (H) -part is $dg\mathbb{Q}W_G H\text{-mod}$, chain complexes of modules over the rational group ring of the Weyl group of H . Note that the homotopy category of $dg\mathbb{Q}W_G H\text{-mod}$ is the category $g\mathbb{Q}W_G H\text{-mod}$. Calculation shows that $\mathcal{E}_a(H)$ has a trivial differential and hence is equal to its homology, furthermore the homology of $\mathcal{E}_t(H)$ is isomorphic to $\mathcal{E}_a(H)$ and is concentrated in degree zero. The intrinsic formality we now use is the fact that a chain complex with homology concentrated in degree zero is equivalent to its homology. Thus by the formal method outlined above, we have the following conclusion.

Theorem 3 (4.3.12) There is a zig-zag of Quillen equivalences between $L_{E_H} G\mathcal{S}_{\mathbb{Q}}$ and $dg\mathbb{Q}W_G H\text{-mod}$. Hence there is a zig-zag of Quillen equivalences between $G\mathcal{S}_{\mathbb{Q}}$ and $dg\mathcal{A}(G) := \prod_{(H) \leq G} dg\mathbb{Q}W_G H\text{-mod}$.

The difference between the above method and that of [SS03b] is the ordering of the work. In our method, we split the category, move to Mackey functors, translate to algebra and then apply a formality result, [SS03b] goes to Mackey functors first, applies formality, moves to algebra and then splits the category. Our formality is that \mathcal{E}_t has homology concentrated in degree zero, [SS03b] uses the fact that \mathcal{E}_{top} has homotopy concentrated in degree zero to replace \mathcal{E}_{top} by $H\mathbf{A}$. This is the same information, just in different contexts.

The result above could have been proved without using the splitting theorem. In which case we would have to use a much larger set of generators: the collection of all smash products of terms $G/H_+ \wedge e_H \mathbb{S}$ as H runs over the conjugacy classes of subgroups of G . Many of these terms would have been rationally contractible, since $G/H_+ \wedge e_H \mathbb{S} \wedge G/K_+ \wedge e_K \mathbb{S}$ is rationally contractible whenever H and K are not conjugate. The splitting removes these extra terms, making the result easier to prove and understand. Thus one of the general advantages of the splitting result is that one can reduce the size of the object set of \mathcal{E}_{top} (and hence of \mathcal{E}_t and \mathcal{E}_a).

The $O(2)$ Case.

We begin in the same place as [Gre98b], we fix W as the group of order two and consider the cofibre sequence $EW_+ \rightarrow S^0 \rightarrow E\widetilde{W}$. We show that EW_+ and $E\widetilde{W}$ satisfy the assumptions of the splitting theorem. We define $\mathcal{CS}_{\mathbb{Q}}$, the model category of cyclic spectra, as the EW_+ -localisation of $O(2)\mathcal{S}_{\mathbb{Q}}$. The model category of dihedral spectra, $\mathcal{DS}_{\mathbb{Q}}$, is defined as the $E\widetilde{W}$ -localisation of $O(2)\mathcal{S}_{\mathbb{Q}}$. Define \mathcal{C} , the set of cyclic subgroups of $O(2)$, to be the closed subgroups of $O(2)$ which are contained in $SO(2)$. All other closed subgroups contain a reflection and we call this set the collection of dihedral subgroups, \mathcal{D} . The reason behind the name cyclic spectra is that the homotopy category of $\mathcal{CS}_{\mathbb{Q}}$ is the homotopy category of $O(2)$ -spectra made from $O(2)$ -cells of the form $O(2)/H_+ \wedge S^n$ with H a cyclic subgroup of $O(2)$ (so $H \leq SO(2)$). Hence we call the remainder, $\mathcal{DS}_{\mathbb{Q}}$, the category of dihedral spectra.

Theorem 4 (6.1.3) There is a strong monoidal Quillen equivalence

$$\Delta : O(2)\mathcal{M}_{\mathbb{Q}} \xleftrightarrow{\quad} \mathcal{CS}_{\mathbb{Q}} \times \mathcal{DS}_{\mathbb{Q}} : \prod.$$

In particular, we have the following natural isomorphism for any G -spectra X and Y

$$[X, Y]_{\mathbb{Q}}^{O(2)} \cong [X \wedge EW_+, Y \wedge EW_+]_{\mathbb{Q}}^{O(2)} \oplus [X \wedge E\widetilde{W}, Y \wedge E\widetilde{W}]_{\mathbb{Q}}^{O(2)}.$$

We expect, by looking at the calculations of [Gre98b], that these two parts behave quite differently and will be classified by quite different methods. The splitting theorem allows us to deal with each part in turn. We expect the dihedral part, $\mathcal{DS}_{\mathbb{Q}}$, to behave somewhat like the finite case, though we do not make this precise or study the dihedral part any further. We have concentrated on the cyclic part, $\mathcal{CS}_{\mathbb{Q}}$. We could try to use the general method to classify $\mathcal{CS}_{\mathbb{Q}}$, but we would still have to find some intrinsic formality argument in order to understand $\mathcal{E}_t(\mathcal{C})$ in terms of some algebraic model. Instead we make precise the relation between $\mathcal{CS}_{\mathbb{Q}}$ and $SO(2)\mathcal{S}_{\mathbb{Q}}$ using the notion of a category with involution. We then intend to make use of the classification of $SO(2)\mathcal{S}_{\mathbb{Q}}$ in [GS] to understand $\mathcal{E}_t(\mathcal{C})$. The motivation here is the relation between $O(2)$ -spaces and $SO(2)$ -spaces: an $O(2)$ -space X is an $SO(2)$ -space with a map $f: X \rightarrow X$, such that $f^2(x) = x$ and $f(tx) = t^{-1}f(x)$ for all $t \in SO(2)$ and $x \in X$ (so f is like a reflection). We formalise this into categorical language, then investigate how this notion can be applied to spectra.

Take a category \mathcal{C} and a functor $\sigma: \mathcal{C} \rightarrow \mathcal{C}$ such that $\sigma^2 = 1$, we call σ an involution and (\mathcal{C}, σ) a category with involution. In our examples all our categories have underlying sets, so it makes sense to use equality. In general one could replace this equality by a fixed natural isomorphism which would form part of the structure. The skewed category of (\mathcal{C}, σ) , denoted $\sigma\#\mathcal{C}$, has objects the maps $f: X \rightarrow \sigma X$ such that $\sigma f \circ f = \text{Id}_X$. Morphisms are pairs $(\alpha, \sigma\alpha)$ making the obvious square commute, for α a morphism of \mathcal{C} . We prove that if \mathcal{C} is a cofibrantly generated model category and σ is a left (and hence right) Quillen functor then $\sigma\#\mathcal{C}$ has a cofibrantly generated model structure. This model structure is defined by the condition that a map $(\alpha, \sigma\alpha)$ is a weak equivalence or fibration exactly when α is in \mathcal{C} .

The simplest example is the case when σ is the identity functor and \mathcal{C} is any category. Here $\text{Id}\#\mathcal{C}$ is the category of W -objects and W -maps in \mathcal{C} (W is the group of order two). Next consider the category of based $SO(2)$ -equivariant spaces $SO(2)\mathcal{T}_*$. The involution is $\sigma = j^*$, pullback along the group homomorphism $j: SO(2) \rightarrow SO(2)$ which takes t to t^{-1} . The skewed category, $j^*\#SO(2)\mathcal{T}_*$, is then the category of $O(2)$ -spaces. Note that the model structure on $j^*\#SO(2)\mathcal{T}_*$ is not the usual model structure on $O(2)$ -spaces, instead a map $f: X \rightarrow Y$ is a weak equivalence if and only if each $f^H: X^H \rightarrow Y^H$ is a weak homotopy equivalence of spaces for all $H \leq SO(2)$. With some work we can construct a similar functor on $SO(2)$ -spectra, so that we have a category with involution $(SO(2)\mathcal{S}_{\mathbb{Q}}, \tau)$.

Theorem 5 (8.2.5) There is a monoidal Quillen equivalence $\tau\#SO(2)\mathcal{S}_{\mathbb{Q}} \xleftrightarrow{\quad} \mathcal{CS}_{\mathbb{Q}}$.

We prove that there is an involution ρ on $\text{mod-}\mathcal{E}_t(SO(2))$ and show that the series of Quillen equivalences between $SO(2)\mathcal{S}_{\mathbb{Q}}$ and $\mathcal{E}_t(SO(2))$ preserves this involution. We can then conclude the following.

Theorem 6 (Corollary 9.4.4) There is a zig-zag of Quillen equivalences between the categories $\tau\#SO(2)\mathcal{S}_{\mathbb{Q}}$ and $\rho\#\text{mod-}\mathcal{E}_t(SO(2))$.

0.5 Applications and Further Work

Cyclic spectra

It remains to complete the classification of $\mathcal{CS}_{\mathbb{Q}}$, which we have identified with the skewed category $\rho\#\text{mod-}\mathcal{E}_t(SO(2))$. From this point the idea is that there should be an involution Υ on $dg\mathcal{A}(SO(2))$ (see Remark 9.4.5). The algebraic model for $\mathcal{CS}_{\mathbb{Q}}$ will then be given by $\Upsilon\#dg\mathcal{A}(SO(2))$. Then if we can show that the zig-zag between $\mathcal{E}_t(SO(2))$ and $\mathcal{E}_a(SO(2))$ of [GS] respects the involutions it should follow formally that $\mathcal{CS}_{\mathbb{Q}}$ and $\Upsilon\#dg\mathcal{A}(SO(2))$ are Quillen equivalent.

Dihedral Spectra

The dihedral part of the $O(2)$ case must still be classified in terms of an algebraic model. The model suggested by [Gre98b] is a category of sheaves over a topological space. Once this is understood one can consider the more general setting of a split short exact sequence $0 \rightarrow SO(2)^n \rightarrow G \rightarrow F \rightarrow 1$ and try understand the F part of the category of G -spectra.

The Homotopy Category of a Skewed Category

For (\mathcal{C}, σ) an involutory model category, we would like to be able to prove an equivalence $\text{Ho}(\sigma\#\mathcal{C}) \simeq \sigma\#\text{Ho}(\mathcal{C})$. We expect that such a result would require $\text{Ho}(\mathcal{C})$ to be rational (so $[X, Y]_{\mathcal{C}}$ is a rational vector space for each X and Y in \mathcal{C}). We would also like to investigate the conditions necessary for the involution to untwist, that is, when is there an equivalence $\sigma\#\text{Ho}(\mathcal{C}) \simeq \text{Ho}(\mathcal{C})^W$? Where the right hand side is the homotopy category of \mathcal{C} with a homotopy action of W , the group of order two. In Proposition 6.1.5 we give [Gre98b, Proposition 3.1 and Corollary 3.2], which prove that $\text{Ho}(\tau\#\mathcal{CS}_{\mathbb{Q}})$ is equivalent to $\text{Ho}(\mathcal{CS}_{\mathbb{Q}})^W$, which is our reason for expecting these results. With suitable assumptions on the nature of \mathcal{C} we hope to prove the results above by following [Gre98b].

Monoidality

The classification outlined above for [GS], the finite case and $O(2)$ is much more compatible with monoidal structures than the work of [Shi02] and [SS03b, Example

5.1.2]. We go through the classification once more and point out which equivalences are monoidal. Begin with $G\mathcal{S}_{\mathbb{Q}}$ and make a more careful choice of generators \mathcal{G}_{top} (the set of generators should be closed under \wedge), then the Quillen equivalence between $G\mathcal{S}_{\mathbb{Q}}$ and $\text{mod-}\mathcal{E}_{top}$ will be symmetric monoidal. Equally, the equivalence between $\text{mod-}\mathcal{E}_a$ and $dg\mathcal{A}(G)$ will be symmetric monoidal provided that the generators \mathcal{G}_a are closed under \otimes . The results of [Shi07b] allow us to construct a category \mathcal{E}_t and $\text{mod-}\mathcal{E}_t$ is Quillen equivalent to $\text{mod-}\mathcal{E}_{top}$. However, due to a technical issue the category $\text{mod-}\mathcal{E}_t$ is not a monoidal category (see Remark 9.3.6). Hence the comparison between $\text{mod-}\mathcal{E}_t$ and $\text{mod-}\mathcal{E}_{top}$ cannot be monoidal, thus, neither can the whole classification. This is the only point of the comparison where monoidality fails; it is probable that this issue can be resolved.

Equivariant Categories

As mentioned in Remark 7.3.17 we identify the notion of a category with involution with a category with an action of order two on it. Thus there is an obvious generalisation to a category with a general group action. So consider a group G , where G is part of a split short exact sequence $0 \rightarrow SO(2)^n \rightarrow G \rightarrow F \rightarrow 1$ with F a finite group. Then we can split the category of G -spectra into a part corresponding to F and a part $G\mathcal{S}_{\mathbb{Q}}(SO(2)^n)$, corresponding to $SO(2)^n$. It should then be possible to describe $G\mathcal{S}_{\mathbb{Q}}(SO(2)^n)$ in terms of the category $SO(2)^n\mathcal{S}_{\mathbb{Q}}$ with an action of F . The theory of these equivariant categories should allow us to classify $G\mathcal{S}_{\mathbb{Q}}(SO(2)^n)$ in terms of $dg\mathcal{A}(SO(2)^n)$ with an action of F on the category. Thus, in this situation, we can extend known classifications and generate new algebraic models from existing ones.

Continuous Quaternions

The next group of interest would be the continuous quaternion group. There is a short exact sequence $1 \rightarrow SO(2) \rightarrow Q_{cts} \rightarrow W \rightarrow 1$ (W the group of order two), but now this sequence is not split. An understanding of how to classify $Q_{cts}\mathcal{S}_{\mathbb{Q}}$ should give a reasonable idea of how to understand G -spectra for a non-split short exact sequence $0 \rightarrow (SO(2))^n \rightarrow G \rightarrow F \rightarrow 1$.

0.6 Organisation of the Thesis

We have divided the thesis into three parts, the first consists of basic notions and the splitting for a general compact Lie group G , the second classifies rational G -spectra for finite G and the third specialises to the case $G = O(2)$ and examines how the splitting theorem and the work of [GS] can be used to study this case.

Part I We begin Chapter 1 with some of the basic notions of model categories, before moving on to describe some of the more technical conditions, especially those which ensure that a monoidal product will behave well on the homotopy category. We give a brief description of the model categories that are used in this thesis and then go into more detail on the categories of equivariant spectra in Section 1.3. We introduce homotopy colimits in Section 1.4 and then use this to construct $S^0\mathbb{Q}$, a rational sphere spectrum, in Section 1.5. Chapter 2 introduces Bousfield localisations of spectra, which we use to make a category of rational spectra by localising at $S^0\mathbb{Q}$. We also construct an equivalent category of rational spectra by considering modules over the ring spectrum $S_{\mathbb{Q}}$ (constructed in Section 1.5). We make further use of Bousfield localisations to prove the splitting theorem for equivariant orthogonal spectra in Chapter 3. We show that this implies the corresponding splitting result for equivariant EKMM spectra and the category of $S_{\mathbb{Q}}$ -modules. We then consider a particular kind of splitting in Section 3.4 that we will use for $O(2)$ -spectra in Chapter 6.

Part II Chapter 4 classifies rational G -spectra for finite G . We begin with the algebraic model which is particularly simple. We then apply the splitting theorem to the category of rational G -spectra and identify the (H) -piece of this splitting with modules over a ring spectrum S_H (there is one piece for each conjugacy class of subgroups (H)). Since we now have a category with every object fibrant, we can proceed through the method of [GS]. Once we have performed the formal parts of this method, we must specialise to our particular case and prove that $\text{mod-}\mathcal{E}_t^H$ is equivalent to $\text{mod-}\mathcal{E}_a^H$ for each conjugacy class of subgroups $(H) \subseteq G$. This is done by studying the structure of the $dg\mathbb{Q}$ -mod-enriched categories \mathcal{E}_t and \mathcal{E}_a . Since this is the first time we use the notion of right modules over an enriched category, we introduce the language and theory of this machinery in Chapter 5.

Part III We consider the group $O(2)$ and see how our splitting theorem can be used to study rational $O(2)$ -spectra. Chapter 6 splits the category into cyclic spectra and dihedral spectra, using the results of Section 3.4. We concentrate on the model category of cyclic spectra, we need to understand how this category is related to $SO(2)$ -spectra, so that we can use the work of [GS] to classify this category in terms of an algebraic model. In Chapter 7 we have abstracted the relation between $O(2)$ -spaces and $SO(2)$ -spaces to define a category with involution and its associated skewed category. We apply this to cyclic spectra to describe this category in terms of the skewed category of rational $SO(2)$ -spectra in Chapter 8. In Chapter 9 we examine the zig-zag of equivalences between rational $SO(2)$ -spectra and $\text{mod-}\mathcal{E}_t(SO(2))$ from [GS]. We

prove that this induces a zig-zag of Quillen equivalences between the skewed category of $\text{mod-}\mathcal{E}_t(SO(2))$ and the category of cyclic spectra. This chapter requires us to use $S_{\mathbb{Q}}$ -modules as our category of rational $SO(2)$ -spectra, since every object of the category must be fibrant to apply the work of [GS].

We include an appendix listing all of the model categories that we use.

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Part I

G-spectra and Localisations

Chapter 1

G -Spectra

We introduce the basic notions necessary to study cohomology theories in a modern setting. We begin with model categories, which make the construction of homotopy categories rigorous and allow us to prove that two homotopy categories are equivalent by checking a small list of conditions. In Section 1.2 we give brief details on the categories that we will use in this thesis. We focus upon the categories of G -spectra in Section 1.3 and go into some details on the properties of these categories. We prove a few well-known results and show some technical model category conditions that we have not been able to find explicitly in the literature. We construct homotopy pushouts and telescopes for G -spectra in Section 1.4 so that we will have definite constructions for our later work. We then make a rational sphere spectrum in Section 1.5 which will be used to define rational G -spectra in Chapter 3. Much of this chapter is definitions and results from other sources, especially so for the first two sections. The new content of Sections 1.3 and 1.4 is mainly in the proofs, which are often considered too standard to be included in the usual sources. The final section is mostly new, though not surprising.

1.1 Model Categories

Many of the results of this thesis are phrased in terms of model categories, which are a general framework for homotopy theory. If one wishes to invert a collection of maps in a category (i.e. formally make them into isomorphisms), one can not always be sure that the result will be a category, model categories are a solution to this problem. Model categories were first introduced in [Qui67], an excellent modern account is [DS95], but we take most of our definitions from Section 1.1 of the comprehensive book [Hov99]. We let d and c be the domain and codomain functors from $\text{Map}\mathcal{C}$ to \mathcal{C} , which exist for any category \mathcal{C} . We give [Hov, Definitions 1.1.1 – 1.1.4] in order.

Definition 1.1.1 Suppose \mathcal{C} is a category

- (i). A map f in \mathcal{C} is a **retract** of a map $g \in \mathcal{C}$ if and only if there is a commutative

diagram of the following form

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ f \downarrow & & g \downarrow & & f \downarrow \\ B & \longrightarrow & D & \longrightarrow & D \end{array}$$

where the horizontal composites are identities.

- (ii). A **functorial factorisation** is an ordered pair (α, β) of functors $\text{Map}\mathcal{C} \rightarrow \text{Map}\mathcal{C}$ such that

$$d \circ \alpha = d, \quad c \circ \alpha = d \circ \beta, \quad c \circ \beta = c, \quad f = \beta(f) \circ \alpha(f)$$

for all $f \in \text{Map}\mathcal{C}$. Hence any commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \downarrow & & \downarrow v \\ C & \xrightarrow{g} & D \end{array}$$

induces a commutative square

$$\begin{array}{ccccc} A & \xrightarrow{\alpha(f)} & (c \circ \alpha)(f) & \xrightarrow{\beta(f)} & B \\ u \downarrow & & (c \circ \alpha)(u, v) \downarrow & & \downarrow v \\ C & \xrightarrow{\alpha(g)} & (c \circ \alpha)(g) & \xrightarrow{\beta(g)} & D \end{array}$$

noting that (u, v) is a morphism in $\text{Map}\mathcal{C}$ between f and g .

Definition 1.1.2 Suppose $i: A \rightarrow B$ and $p: X \rightarrow Y$ are maps in a category \mathcal{C} . Then i has the **left lifting property with respect to** p and p has the **right lifting property with respect to** i if, for every commutative diagram of the following form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

there is a lift $h: B \rightarrow X$ such that $hi = f$ and $ph = g$.

Definition 1.1.3 A **model structure** on a category \mathcal{C} is three subcategories of \mathcal{C} called weak equivalences, cofibrations and fibrations and two functorial factorisations (α, β) and (γ, δ) satisfying the following properties:

- (i). If f and g are morphisms of \mathcal{C} such that gf is defined and two of f , g and gf are weak equivalences, then so is the third.

- (ii). If f and g are morphisms of \mathcal{C} such that f is a retract of g and g is a weak equivalence, cofibration or fibration, then so is f .
- (iii). Define a map to be an **acyclic cofibration** if it is both a cofibration and a weak equivalence. Similarly, define a map to be an **acyclic fibration** if it is both a fibration and a weak equivalence. Then acyclic cofibrations have the left lifting property with respect to fibrations and cofibrations have the left lifting property with respect to acyclic fibrations.
- (iv). For any morphism f , $\alpha(f)$ is a cofibration, $\beta(f)$ is an acyclic fibration, $\gamma(f)$ is an acyclic cofibration and $\delta(f)$ is a fibration.

Definition 1.1.4 A **model category** is a category \mathcal{C} with all small limits and colimits and a model structure on \mathcal{C} .

When we need to emphasise the properties of maps in diagrams we will use the following shorthand: $X \xrightarrow{\sim} Y$ for the weak equivalences, $X \xrightarrow{\twoheadrightarrow} Y$ for the cofibrations and $X \xrightarrow{\twoheadrightarrow} Y$ for the fibrations. Since a model category has all small limits and colimits it has an initial object \emptyset and a terminal object $*$, these are the colimit and limit of the empty diagram respectively. So for any object X there is a unique pair of maps $\emptyset \rightarrow X \rightarrow *$. We call X **cofibrant** if the map $\emptyset \rightarrow X$ is a cofibration, similarly X is called **fibrant** if the map $X \rightarrow *$ is a fibration. By using the factorisation axioms one can take the map $\emptyset \rightarrow X$ and factor it into a cofibration followed by an acyclic fibration: $\emptyset \rightarrow \widehat{C}X \rightarrow X$. We call this process cofibrant replacement. The functorial factorisations ensure that \widehat{C} is a functor. We can perform the equivalent construction for the map $X \rightarrow *$ and we obtain \widehat{F} the fibrant replacement functor. We will decorate this notation where necessary to indicate which model structure we are considering. If the canonical map $\emptyset \rightarrow *$ is an isomorphism we call the model category **pointed**.

We now cut to the chase and give a rough and ready theorem stating the existence of a homotopy category. The proof of this theorem (that is, the construction of the homotopy category) is a little involved and we leave it to the excellent accounts of model categories that we have already mentioned.

Theorem 1.1.5 *If \mathcal{C} is a model category, then there is a category $\mathrm{Ho}\mathcal{C}$, called the **homotopy category** of \mathcal{C} with a functor $\gamma: \mathcal{C} \rightarrow \mathrm{Ho}\mathcal{C}$ such that γf is an isomorphism if and only if f is a weak equivalence. Furthermore if $F: \mathcal{C} \rightarrow \mathcal{D}$ is any functor which takes every weak equivalence of \mathcal{C} to an isomorphism of \mathcal{D} then there is a unique functor $\mathrm{Ho} F: \mathrm{Ho}\mathcal{C} \rightarrow \mathcal{D}$ such that $\mathrm{Ho} F \circ \gamma = F$.*

Now we move on to [Hov99, Section 1.3] and introduce the language necessary to compare model categories and the notion of equivalent model categories. The definition below is [Hov99, Definition 1.3.1] and the following lemma is [Hov99, Lemma 1.3.10].

Definition 1.1.6 If \mathcal{C} and \mathcal{D} are model categories then a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a **left Quillen functor** if F preserves cofibrations and acyclic cofibrations. Similarly F is a **right Quillen functor** if F preserves fibrations and acyclic fibrations. An adjoint

pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a **Quillen pair** if either the left adjoint is a left Quillen functor, or equivalently the right adjoint is a right Quillen functor.

Lemma 1.1.7 *A left Quillen functor $F : \mathcal{C} \rightarrow \mathcal{D}$ passes to a functor $LF : \mathrm{Ho} \mathcal{C} \rightarrow \mathrm{Ho} \mathcal{D}$, similarly a right Quillen functor $G : \mathcal{D} \rightarrow \mathcal{C}$ passes to a functor $RG : \mathrm{Ho} \mathcal{D} \rightarrow \mathrm{Ho} \mathcal{C}$. A Quillen pair induces an adjunction $LF : \mathrm{Ho} \mathcal{C} \rightleftarrows \mathrm{Ho} \mathcal{D} : RG$.*

We now give [Hov99, Definition 1.3.12 and Proposition 1.3.13].

Definition 1.1.8 A Quillen pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a **Quillen equivalence** if, for all cofibrant X in \mathcal{C} and all fibrant Y in \mathcal{D} , a map $f : X \rightarrow GY$ is a weak equivalence of \mathcal{C} if and only if $\tilde{f} : FX \rightarrow Y$ is a weak equivalence of \mathcal{D} .

Proposition 1.1.9 *A Quillen pair (F, G) is a Quillen equivalence if and only if the adjoint pair (LF, RG) is an equivalence of categories.*

This result is why we use model categories, it allows us to compare homotopy categories by checking a relatively simple criterion. Now we introduce some more structure, that of a monoidal product (such as the smash product of spaces).

Definition 1.1.10 An **adjunction of two variables** ([Hov99, Definition 4.2.12]) $\otimes : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$, is a functor \otimes with two ‘right adjoints’ $\mathrm{Hom}_r : \mathcal{D}^{op} \times \mathcal{E} \rightarrow \mathcal{C}$ and $\mathrm{Hom}_l : \mathcal{C}^{op} \times \mathcal{E} \rightarrow \mathcal{D}$. A **Quillen bifunctor** ([Hov99, Definition 4.2.1]) is an adjunction of two variables \otimes such that for cofibrations $f : U \rightarrow V$ in \mathcal{C} and $g : W \rightarrow X$ in \mathcal{D} the induced map (called the **pushout product**)

$$f \square g : V \otimes W \coprod_{U \otimes W} U \otimes X \longrightarrow V \otimes X$$

is a cofibration of \mathcal{E} which is a weak equivalence when one of f or g is.

Now we give [Hov99, Definition 4.2.6].

Definition 1.1.11 A **monoidal model category**, \mathcal{C} , is a monoidal category $(\mathcal{C}, \otimes, \mathbb{S})$ that is a model category such that $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a Quillen bifunctor and for any cofibrant replacement of the unit $q : \hat{c}S \rightarrow S$ the natural map $\hat{c}S \otimes X \rightarrow S \otimes X$ is a weak equivalence for all cofibrant X . These two requirements are also known as the **pushout product axiom**.

Proposition 1.1.12 *The closed monoidal product on a monoidal model category passes to a closed monoidal product on the homotopy category.*

This proposition is part of [Hov99, Theorem 4.3.2]. Often we have a monoidal product that is symmetric, for this we use [Hov, Definition 4.1.4] below.

Definition 1.1.13 A **symmetric monoidal model category** is a monoidal category \mathcal{C} with a natural commutativity isomorphism $T_{X,Y} : X \otimes Y \rightarrow Y \otimes X$. We require T to be compatible with the unit isomorphisms and associativity of \mathcal{C} and satisfy $T_{Y,X} \circ T_{X,Y} = \mathrm{Id}_{X \otimes Y}$.

Now we want to know when this extra structure is preserved by Quillen functors. The following two definitions are [SS03a, Definitions 3.3 and 3.6].

Definition 1.1.14 A **monoidal functor** is a functor between monoidal categories $F: \mathcal{C} \rightarrow \mathcal{D}$ with a morphism $\nu: S_{\mathcal{D}} \rightarrow F(S_{\mathcal{C}})$ and natural morphisms $FX \otimes FY \rightarrow F(X \otimes Y)$ which are coherently associative and unital. If these maps are isomorphisms then F is a **strong monoidal functor**. A **symmetric monoidal functor** between symmetric monoidal categories is a monoidal functor F such that the following diagram commutes.

$$\begin{array}{ccc} FX \otimes FY & \longrightarrow & F(X \otimes Y) \\ T_{FX, FY} \downarrow & & F(T_{X, Y}) \downarrow \\ FY \otimes FX & \longrightarrow & F(Y \otimes X) \end{array}$$

Definition 1.1.15 A **monoidal Quillen pair** is a Quillen pair $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ between monoidal model categories with R a monoidal functor such that the following conditions hold.

- (i). For all cofibrant objects A and B in \mathcal{C} the induced map (see below) $L(A \otimes_{\mathcal{C}} B) \rightarrow LA \otimes_{\mathcal{D}} LB$ is a weak equivalence in \mathcal{D} .
- (ii). For some (hence any) cofibrant replacement of the unit in \mathcal{C} , $\widehat{c}S_{\mathcal{C}} \rightarrow S_{\mathcal{C}}$, the composite $L\widehat{c}S_{\mathcal{C}} \rightarrow LS_{\mathcal{C}} \rightarrow S_{\mathcal{D}}$ is a weak equivalence in \mathcal{D} .

The map $L(A \otimes_{\mathcal{C}} B) \rightarrow LA \otimes_{\mathcal{D}} LB$ is the adjoint of the composite: $A \otimes_{\mathcal{C}} B \rightarrow RLA \otimes_{\mathcal{C}} RLB \rightarrow R(LA \otimes_{\mathcal{D}} LB)$.

A **strong monoidal Quillen pair** is a monoidal Quillen pair such that the maps $L(A \otimes_{\mathcal{C}} B) \rightarrow LA \otimes_{\mathcal{D}} LB$ and $LS_{\mathcal{C}} \rightarrow S_{\mathcal{D}}$ are isomorphisms, thus L is a strong monoidal functor. A monoidal Quillen pair is a **symmetric monoidal Quillen pair** if the right adjoint is a symmetric monoidal functor and the following diagram commutes.

$$\begin{array}{ccc} L(X \otimes Y) & \longrightarrow & LX \otimes LY \\ L(T_{X, Y}) \downarrow & & T_{LX, LY} \downarrow \\ L(Y \otimes X) & \longrightarrow & LY \otimes LX \end{array}$$

Thus a strong monoidal adjunction is precisely the same as a monoidal Quillen adjunction of [Hov99, Definition 4.2.16]. The conditions on the left adjoint of a monoidal Quillen pair ensure that this passes to a strong monoidal functor on the homotopy categories. We have the result below.

Proposition 1.1.16 *A monoidal Quillen pair gives a strong monoidal adjunction between the homotopy categories.*

A monoidal Quillen equivalence between a pair of monoidal model categories tells us that they have the same homotopy categories and that we can use this adjunction

to compare the monoidal products. In particular, there is [SS03a, Theorem 3.12], very roughly this says that if you add in some extra assumptions then a monoidal Quillen equivalence passes to Quillen equivalences between categories of modules and monoids. A strong monoidal Quillen equivalence behaves better from a categorical point of view. For example, it will preserve structures such as enrichments, tensorings and cotensorings.

We need a technical condition on our model categories to ensure good behaviour. This is of great importance when constructing new model categories from existing ones. The following definition is [Hov99, Definition 2.1.17], we will explain the new concepts in the definition shortly.

Definition 1.1.17 A **cofibrantly generated** model category is a model category \mathcal{C} with sets of maps I and J such that:

- (i). the domains of I are small relative to I -cell,
- (ii). the domains of J are small relative to J -cell,
- (iii). the class of fibrations is the class of maps with the right lifting property with respect to every map in J ,
- (iv). the class of acyclic fibrations is the class of maps with the right lifting property with respect to every map in I .

We call I the set of **generating cofibrations** and J the set of **generating acyclic cofibrations**.

For λ an ordinal, a λ -sequence in \mathcal{C} (a category with all small colimits) is a colimit preserving functor $X: \lambda \rightarrow \mathcal{C}$. If κ is a cardinal, an ordinal λ is κ -filtered if it is a limit ordinal and if $A \subseteq \lambda$ with $|A| \leq \kappa$, then $\sup A < \lambda$. This is from [Hov99, Definitions 2.1.1 and 2.1.2], the following pair of definitions are [Hov99, Definitions 2.1.3 and 2.1.9].

Definition 1.1.18 Let \mathcal{C} be a category with all small colimits, I a collection of maps in \mathcal{C} and κ a cardinal. Then an object A of \mathcal{C} is **κ -small with respect to I** if the map of sets

$$\mathrm{Colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \longrightarrow \mathcal{C}(A, \mathrm{Colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism for all κ -filtered ordinals λ and all λ -sequences X such that $X_\beta \rightarrow X_{\beta+1}$ is in \mathcal{D} . We say that A is **small with respect to I** if there is an cardinal κ such that A is κ -small with respect to I . We say that A is **small** if it is small with respect to \mathcal{C} itself.

Definition 1.1.19 For a set of maps I in a model category the collection I -cell of **relative I -cell complexes** is the collection of transfinite compositions of pushouts of maps of I . That is a map in I -cell has form $X_0 \rightarrow \mathrm{Colim}_{\beta < \lambda} X_\beta$ for some ordinal λ

and some λ sequence X , such that for each $\beta + 1 < \lambda$ there is a pushout square as below such that g_β is in I .

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ g_\beta \downarrow & \lrcorner & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

The definition and development below are taken from [SS03b], which uses the language of triangulated categories, which we do not define. Instead, we note that the homotopy categories of all of the model categories that we will use are triangulated categories. In the language of [SS03b, Definition 2.1.1] these categories are stable model categories. In particular each of these has a ‘shift’ functor, a pair of inverse equivalences of the homotopy category (Σ, Ω) . For spectra this is the suspension and loop pair, for chain complexes of R modules the adjoint pair is $(-\otimes_R R[1], \text{Hom}_R(R[1], -))$. For $n \geq 0$ we let Σ^n be the n -fold composite of Σ and Σ^{-n} be the n -fold composite of Ω . The definition below is [SS03b, Definition 2.1.2].

Definition 1.1.20 An object X of a triangulated category \mathcal{T} is **compact** if

$$[X, \bigvee_i Y_i] \cong \bigoplus_i [X, Y_i]$$

for any collection of objects Y_i . A full triangulated subcategory of \mathcal{T} is called a **localising subcategory** if it is closed under coproducts in \mathcal{T} . A set \mathcal{P} , of objects of \mathcal{T} , is called a set of **generators** if the only localising subcategory of \mathcal{T} that contains \mathcal{P} is \mathcal{T} itself.

We will say that an object X of a stable model category is compact or a generator if it is so in the homotopy category. In the homotopy category of a stable model category we can define $[X, Y]_*$, the **graded set of maps** in the homotopy category from X to Y to be the graded set with $[X, Y]_n = [\Sigma^n X, Y]$ for $n \in \mathbb{Z}$.

Lemma 1.1.21 *Let \mathcal{T} be a triangulated category with infinite coproducts and let \mathcal{P} be a set of compact objects. Then the following are equivalent:*

- (i). *The set \mathcal{P} is a set of generators.*
- (ii). *An object X of \mathcal{T} is acyclic if and only if $[P, X]_* = 0$ for all $P \in \mathcal{P}$.*

Proof This is [SS03b, Lemma 2.2.1]. ■

The following is [SS00, Definition 3.3], this is an important condition which allows one (see [SS00, Theorem 4.1]) to make model structures for categories of modules and algebras over a ring object.

Definition 1.1.22 Let \mathcal{C} be a monoidal model category, let P be the class of maps of the form $f \otimes \text{Id}_Z: X \otimes Z \rightarrow Y \otimes Z$ where f is an acyclic cofibration and Z is an object of \mathcal{C} . The model category \mathcal{C} satisfies the **monoid axiom** if every map in P -cell is a weak equivalence.

1.2 Spaces, Spectra and Equivariance

We will use many different categories, as is common in algebraic topology. We introduce some of the more important ones here. All of our examples are pointed model categories. We reiterate that throughout G will be a compact Lie group.

Spaces The category of spaces we use is \mathcal{T}_* , compactly generated weak Hausdorff based spaces. Based means that every object of the category has a distinguished point, called the basepoint. Maps of based spaces are required to preserve the basepoints. This is a symmetric monoidal category under the smash product. The model structure is cofibrantly generated with generating cofibrations the boundary inclusions $S_+^{n-1} \rightarrow D_+^n$ and generating acyclic cofibrations the inclusions $D_+^n \rightarrow (D^n \times I)_+$, for $n \geq 0$. The weak equivalences are the weak homotopy equivalences (π_* -isomorphisms) and the fibrations are the Serre fibrations: those maps with the right lifting property with respect to the generating acyclic cofibrations. This material has been taken from [Hov99, Section 2.4]. In general whenever we mention a topological space we mean an object of \mathcal{T}_* .

***G*-Spaces** We can also consider $G\mathcal{T}_*$, based G -equivariant spaces, as in [MM02, Chapter III, Section 1]. The basepoint of a space is required to be G -fixed. The generating cofibrations are those maps $(G/H \times S^{n-1})_+ \rightarrow (G/H \times D^n)_+$ where $n \geq 0$ and H runs through the closed subgroups of G . The generating acyclic cofibrations are the maps $(G/H \times D^n)_+ \rightarrow (G/H \times D^n \times I)_+$. A map f of G -spaces is a weak equivalence or fibration if and only if each f^H (the map induced on H -fixed points) is a weak equivalence or fibration of topological spaces. This category is symmetric monoidal under the smash product of G -spaces, where G acts diagonally.

Simplicial Sets We take the following information from [Hov99, Chapter 3]. Let Δ be the category with objects $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and morphisms the set maps such that $x \geq y$ implies $f(x) \geq f(y)$. A **simplicial set** is a contravariant functor $\Delta \rightarrow \mathbf{Set}$. This is the category of simplicial sets and we will denote the category of based simplicial sets by \mathbf{sSet} . The generating cofibrations are the maps $\partial\Delta[n] \rightarrow \Delta[n]$ for $n \geq 0$. The generating acyclic cofibrations are $\Lambda^r[n] \rightarrow \Delta[n]$ for $n > 0$ and $0 \leq r \leq n$. The cofibrations are the injective maps and the fibrations are precisely the Kan fibrations. The weak equivalences are those maps f such that the geometric realisation $|f|$ is a weak homotopy equivalence of topological spaces. We must also use $\mathbf{sQ-mod}$, simplicial \mathbb{Q} -modules. This is the category of contravariant functors $\Delta \rightarrow \mathbb{Q-mod}$. The model structure on this category has fibrations and weak equivalences defined in terms of underlying simplicial sets.

Symmetric Spectra We take our definitions of symmetric spectra from [HSS00]. A **symmetric spectrum** is a collection $\{X_n\}_{n \geq 0}$ of based simplicial sets with a basepoint preserving left action of Σ_n on X_n .

This collection must have based maps $S^1 \wedge X_n \rightarrow X_{n+1}$ for each $n \geq 0$ such that the map $S^p \wedge X_n \rightarrow X_{n+p}$ is $\Sigma_p \times \Sigma_n$ -equivariant for $p \geq 1$ and $n \geq 0$ (with Σ_p permuting the p -factors of $S^p = (S^1)^p$). A map $f: X \rightarrow Y$ of symmetric spectra is then a sequence of pointed Σ_n -equivariant maps $f_n: X_n \rightarrow Y_n$ which commute with the structure maps. There is a tensor product \otimes of symmetric spectra X and Y defined by

$$(X \otimes Y)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q).$$

Now we consider a symmetric spectrum S with $S_n = S^n$ (with the permutation action of Σ_n on S_n), this is a ring object with respect to \otimes . A symmetric spectrum is naturally a left S -module. The smash product of symmetric spectra is then given by $X \otimes_S Y$, the tensor product over S . A map is a cofibration if it has the left lifting property with respect to those maps f such that each f_n is an acyclic fibration of simplicial sets. The fibrations are characterised by the right lifting property and the fibrant objects are the Ω -spectra, those spectra such that the adjoints of the structure maps $(S^1 \wedge X_n \rightarrow X_{n+1})$ are weak equivalences. A spectrum E is injective if given $f: X \rightarrow Y$, a monomorphism and a level-wise equivalence and a map $g: X \rightarrow E$ there is an $h: Y \rightarrow E$ such that $hf = g$. The weak equivalences are those maps f such that $E^0 f$ is an isomorphism for every injective Ω -spectrum E . The notation for this category is $Sp^\Sigma(\mathbf{sSet})$ which we will usually shorten to Sp^Σ .

There are also versions of symmetric spectra defined in terms of topological spaces as constructed in [MMSS01]. In that paper they also consider a model structure Sp_+^Σ called the positive stable model structure. Here the fibrant objects are those spectra which are Ω -spectra in all levels greater than 0 and the weak equivalences are defined in terms of injective positive Ω -spectra. The paper [Hov01] generalises these results still further and considers symmetric spectra in general model categories. Fix some monoidal model category \mathcal{C} and choose some cofibrant object K to take the place of S^1 . The construction of symmetric spectra in \mathcal{C} then follows the same plan as above, replacing simplicial sets with \mathcal{C} and S^1 with K . This category will be denoted $Sp^\Sigma(\mathcal{C}; K)$ or just $Sp^\Sigma(\mathcal{C})$.

Orthogonal Spectra We take this from [MM02, Chapter II]. Choose some collection of finite dimensional irreducible real representations of G . We let U be the direct sum of countably many copies of each of these representations, we say that U is a **universe**. If we take every finite dimensional representation in our collection then we call U **complete**. Now we define an **indexing G -space in U** to be a finite dimensional sub G -inner product space of U .

Define \mathcal{I}_G^U to be a category with objects those real inner G -product spaces isomorphic to an indexing G -space in U . The morphisms of this category are the G -spaces of (non-equivariant) linear isometric isomorphisms, with G acting by conjugation. An \mathcal{I}_G -**space** is then a functor, enriched over based G -spaces, from \mathcal{I}_G^U to based G -spaces. Let $G\mathcal{I}\mathcal{T}$ be the category of \mathcal{I}_G -spaces with morphisms the G -equivariant maps. There is then an obvious notion of an external smash product of two \mathcal{I}_G -spaces X and Y : $X \bar{\wedge} Y: \mathcal{I}_G^U \times \mathcal{I}_G^U \rightarrow G\mathcal{I}\mathcal{T}$. Using a left Kan extension one can internalise

this and obtain a smash product operation on \mathcal{I}_G -spaces. The category of **orthogonal G -spectra**, $G\mathcal{I}\mathcal{S}$ is the category of left \mathbb{S} -modules in $G\mathcal{I}\mathcal{T}$, where \mathbb{S} is the \mathcal{I}_G -space which takes V to S^V , called the sphere spectrum. The monoidal product on orthogonal G -spectra is the smash product over \mathbb{S} , $X \wedge_{\mathbb{S}} Y := \text{Coeq}(X \wedge \mathbb{S} \wedge Y \rightrightarrows X \wedge Y)$.

EKMM S -modules We also make use of the category of G -equivariant **EKMM S -modules**. The brief details that we give are taken from [MM02, Chapter IV], which takes the work of [EKMM97] and adapts it to the equivariant setting. We start with the category of G -prespectra. An object of this category is a collection of G -spaces $X(V)$, for each indexing space V of some G -universe U with G -equivariant structure maps $S^{W-V} \wedge X(V) \rightarrow X(W)$ that satisfy the obvious transitivity condition for $V \subset W \subset X$. A map of G -prespectra is a collection of equivariant maps $f(V): X(V) \rightarrow Y(V)$ that commute with the structure maps. A G -May spectrum is a prespectrum such the adjoints of the structure maps, $X(V) \rightarrow F(S^{W-V}, X(W))$, are homeomorphisms. Any G -prespectrum can be made into a G -May spectrum, this is called spectrification.

Let $\mathcal{I}(U, U)$ be the space of linear isometries $U \rightarrow U$, with G acting by conjugation. Now recall the notion of the half-twisted smash product from [EKMM97]. This construction can be applied to the equivariant case and for a G -May spectrum X we have a G -May spectrum $\mathcal{I}(U, U) \ltimes X$. There is a monad $\mathbb{L} = \mathcal{I}(U, U) \ltimes (-)$ on G -May spectra with G -equivariant structure maps. We then consider \mathbb{L} -spectra: the category of modules over \mathbb{L} , where the action map is required to be a G -map. The category of \mathbb{L} -spectra has an associative and commutative smash product $\wedge_{\mathcal{I}}$. The sphere May-spectrum S (the spectrification of the prespectrum which takes value S^V at level V) is an \mathbb{L} -spectrum. If N is an \mathbb{L} -spectrum there is a natural map of \mathbb{L} -spectra $\lambda_N: S \wedge_{\mathcal{I}} N \rightarrow N$. The category of S -modules is defined to be those \mathbb{L} -spectra N such that λ_N is an isomorphism. This is our desired category of EKMM S -modules, written GM . Let V be an indexing space and H a subgroup of G , then a generalised sphere spectrum is a spectrum of the form $S \wedge_{\mathcal{I}} \mathbb{L}\Sigma_V^{\infty}(G/H_+ \wedge S^n)$ for $n \geq 0$. These are used to define the model structure on GM can be thought of the building blocks of the category.

Sometimes we will use the generic term spectrum to indicate either of an orthogonal spectrum or an EKMM S -module. We will do so when either the category is implicitly understood or when we are making a statement that applies to either of the above categories.

Differential graded R -Modules It is appropriate to define the usual model structure on chain complexes of R -modules, $dgR\text{-mod}$, in this section. This structure is known as the **projective model structure**. We take these definitions and results from [Hov99, Section 2.3]. A map of chain complexes is a weak equivalence if it is a homology isomorphism and a fibration if it is a surjection. Cofibrations are level-wise split monomorphisms with cofibrant cokernel. For each $n \in \mathbb{Z}$, let $S^n R$ be the chain complex concentrated in degree n , where it takes value R and let $D^n R$ be the chain complex with R in degrees n and $n - 1$ and zeroes elsewhere, with the identity as the differential from degree n to $n - 1$. This is cofibrantly generated model category with

generating cofibrations the maps $S^{n-1}R \rightarrow D^n R$ and generating acyclic cofibrations $0 \rightarrow D^n R$. We will also need to consider $dgR\text{-mod}_+$, the category of non-negatively graded chain complexes of R -modules. This has a model structure with weak equivalences the homology isomorphisms and fibrations the surjections.

1.3 The categories $G\mathcal{I}\mathcal{S}$ and GM

Orthogonal spectra and EKMM S -modules are the most important of the categories that we will consider. We give some basic results that will be of use later. Both of these categories have forgetful functors to the category of G -prespectra. The weak equivalences of orthogonal spectra and EKMM S -modules are defined in terms of these underlying prespectra. The following is [MM02, Chapter III, Definition 3.2].

Definition 1.3.1 For H a subgroup of G and r an integer, the homotopy group $\pi_r^H(X)$ of a G -prespectrum X is

$$\begin{aligned} \pi_r^H(X) &= \operatorname{Colim}_V \pi_r^H(\Omega^V X(V)) & \text{if } r \geq 0 \\ \pi_{-r}^H(X) &= \operatorname{Colim}_{V \supset \mathbb{R}^r} \pi_0^H(\Omega^{V-\mathbb{R}^r} X(V)) & \text{if } r > 0 \end{aligned}$$

where the colimits run over the G -indexing spaces in U .

Theorem 1.3.2 For G a compact Lie group, the following classes of maps define a cofibrantly generated, proper, closed symmetric monoidal model structure on $G\mathcal{I}\mathcal{S}$, the category of G -equivariant orthogonal spectra. This model structure satisfies the monoid axiom. The weak equivalences are those maps f such that $\pi_r^H(f)$ is an isomorphism for all subgroups H of G and all integers r . The cofibrations are the maps with the left lifting property with respect to maps which are level-wise acyclic fibrations of G -spaces. The fibrations are those maps with the right lifting property with respect to cofibrations which are also weak equivalences.

Proof This summary consists of [MM02, Chapter III, Theorem 4.2 and Proposition 7.5]. ■

As one would expect, we will often call weak equivalences π_* -isomorphisms. We will shortly define a model structure on EKMM spectra and we would like a strong monoidal Quillen equivalence between $G\mathcal{I}\mathcal{S}$ and EKMM spectra. To obtain such an equivalence we need a slightly different model structure on orthogonal spectra called the **positive model structure** (see [MM02, Chapter III, Section 5]). The other reason to use the positive model structure is to create a model structure of commutative ring spectra in orthogonal spectra. One cannot do this with the usual model structure on orthogonal spectra as is commented upon in [MMSS01, Section 14] which references [Lew91].

A positive level acyclic fibration is a map of orthogonal spectra that is an acyclic fibration of G -spaces on all levels V with $V^G \neq 0$. A positive cofibration is a map with the left lifting property with respect to the positive level acyclic fibrations. A positive fibration is a map with the right lifting property with respect to the positive cofibrations that are also π_* -isomorphisms.

Theorem 1.3.3 *The positive cofibrations, positive fibrations and weak equivalences define a cofibrantly generated, proper, closed symmetric monoidal model structure on G -equivariant orthogonal spectra. This model category satisfies the monoid axiom and will be denoted by $G\mathcal{J}\mathcal{S}_+$. The identity functor is the left adjoint of a Quillen equivalence $\text{Id}: G\mathcal{J}\mathcal{S}_+ \rightarrow G\mathcal{J}\mathcal{S}$.*

Proof This is [MM02, Chapter III, Theorem 5.3 and Propositions 5.8 and 7.3]. ■

We now turn to EKMM S -modules, the model structure we are interested in is the generalised cellular structure. The generating cofibrations are the maps $E \rightarrow I \wedge E$ and the generating acyclic cofibrations are the maps $(I \wedge E) \rightarrow (I \wedge E) \wedge I_+$ for a generalised sphere spectrum.

Theorem 1.3.4 *For G a compact Lie group, the π_* -isomorphisms, generalised cofibrations and restricted q -fibrations form a cofibrantly generated, proper, closed symmetric monoidal model structure on GM . There is a strong symmetric monoidal Quillen equivalence*

$$\mathbb{N}: G\mathcal{J}\mathcal{S}_+ \rightleftarrows GM: \mathbb{N}^\#.$$

Proof We have taken this from [MM02, Chapter IV, Theorems 1.1, 1.2, and 2.9]. ■

We remind the reader that the categories $G\mathcal{J}\mathcal{S}$, $G\mathcal{J}\mathcal{S}_+$ and GM depend on the choice of universe U . When we wish to specify the universe we are working with, we shall decorate the notation for these categories accordingly. We will always require that our universe is complete. We will (temporarily) denote the sphere spectrum in each of these categories by \mathbb{S} .

Definition 1.3.5 For G a compact Lie group the **Burnside ring** $A(G)$ is defined to be $[\mathbb{S}, \mathbb{S}]^G$, maps in the homotopy category of GM or $G\mathcal{J}\mathcal{S}$.

See [LMSM86, Chapter V, Definitions 2.1 and 2.9] for more details. The ring multiplication is given by composition, which is commutative since $[f \circ g] = [f \wedge g] = [g \wedge f]$, for homotopy classes of maps $[f]$ and $[g]$. It is a well known result that when G is finite, $[\mathbb{S}, \mathbb{S}]^G$ is isomorphic as a ring to the Grothendieck group of isomorphism classes of finite G -sets. The following result implies that $\pi_0^G(\mathbb{S}) \cong A(G)$.

Lemma 1.3.6 *Let X be an orthogonal spectrum, then for any subgroup H of G and integers $q \geq 0$, $p > 0$*

$$\begin{aligned} [\Sigma^q S^0 \wedge G/H_+, X]^G &\cong \pi_q^H(X) \\ [F_p S^0 \wedge G/H_+, X]^G &\cong \pi_{-p}^H(X) \end{aligned}$$

where the left hand side denotes maps in the homotopy category of $G\mathcal{J}\mathcal{S}$.

Proof We apply [MM02, Chapter III, Theorem 4.16] which expresses maps of orthogonal spectra in terms of their underlying prespectra. Then [Ada74, Part III, Proposition 2.8] relates maps in the homotopy category of G -prespectra to homotopy groups. ■

The same result holds of GM , the only change necessary is to use [MM02, Chapter IV, Theorem 2.9] to move to prespectra.

Definition 1.3.7 For a subgroup H of G , the inclusion of H in G will be written ι_H . The map $G \rightarrow \{e\}$ will be written ε_G .

From these maps we have the change of groups functors ι_H^* and ε_G^* . For X a G -space, $\iota_H^*(X)$ is X considered as an H -space. A non-equivariant space Y can be thought of as a G -space with trivial action, we call this G -space $\varepsilon_G^*(Y)$, this functor is known as the inflation functor.

Lemma 1.3.8 *There is a Quillen pair $\varepsilon_G^* : \mathcal{T}_* \rightleftarrows G\mathcal{T}_* : (-)^G$ and for each subgroup H of G there is a Quillen pair $G_+ \wedge_H (-) : H\mathcal{T}_* \rightleftarrows G\mathcal{T}_* : \iota_H^*$. Furthermore these functors are related by the natural isomorphism $\varepsilon_H^* \cong \iota_H^* \circ \varepsilon_G^*$.*

There are spectrum level versions of these functors for both $G\mathcal{J}\mathcal{S}$ and GM . The definitions differ of course, but the essential idea comes from the space level versions. More details can be found in [MM02, Chapter V, Sections 2 and 3] for $G\mathcal{J}\mathcal{S}$ and [MM02, Chapter VI, Sections 1 and 3] for GM . Since we are working equivariantly we need to adjust our notion of compact slightly.

Definition 1.3.9 An object X of $G\mathcal{J}\mathcal{S}$ or GM is **H -compact** if

$$[\iota_H^* X, \bigvee_i Y_i]^H \cong \bigoplus_i [\iota_H^* X, Y_i]^H$$

for any collection of H -spectra Y_i .

That is, X is H -compact if $\iota_H^* X$ is compact in the category of H -spectra.

Lemma 1.3.10 *The suspension spectra $\Sigma^\infty G/H_+$ for H a closed subgroup of G are a set of G -compact generators for the category of G -spectra. Hence \mathbb{S} is H -compact for each subgroup H of G .*

Proof This is a well known fact, see [LMSM86, Chapter I, Definition 4.4 and Lemma 5.3] or [HPS97, Theorem 9.4.3]. We prove this result here, though we will need some definitions and results from later in the work. That the G/H_+ are generators follows from Lemma 1.3.6. That these objects are G -compact follows from Proposition 3.1.5. Take some collection $\{Y_i\}_{i \in I}$, we describe $\bigvee_{i \in I} Y_i$ in terms of a filtered colimit. Consider the diagram $\mathbb{P}(I)$ with object set the collection of subsets of I and morphisms the inclusions. This is obviously a filtered diagram and if we define a functor $X : \mathbb{P}(I) \rightarrow G\mathcal{J}\mathcal{S}$ by $X(J) = \bigvee_{j \in J} Y_j$ with morphisms the obvious inclusions we see that $\text{Colim}_J X(J) \cong \bigvee_{i \in I} Y_i$. Furthermore the maps $X(J) \rightarrow X(J')$ for $J \subseteq J'$ are all h -cofibrations (Definition 1.3.13), so that $\text{Colim}_J X(J)$ is weakly equivalent to $\text{HoColim}_J X(J)$ (Definition 1.4.6). Hence we have the following isomorphisms which complete the proof.

$$\pi_*^H(\bigvee_{i \in I} Y_i) \cong \pi_*^H(\text{Colim}_J X(J)) \cong \text{Colim}_J \pi_*^H(X(J)) \cong \bigoplus_{i \in I} \pi_*^H(Y_i)$$

■

For the result below we need a definition: the G -space $\mathcal{F}G$ is the collection of subgroups of G with finite index in their normaliser with topology given by the Hausdorff topology.

Lemma 1.3.11 (tom Dieck) *For G a compact Lie group, there is an isomorphism of rings $A(G) \otimes \mathbb{Q} \cong C(\mathcal{F}G/G, \mathbb{Q})$.*

Proof We have taken this result from [LMSM86, Chapter V, Lemma 2.10], which references [tD77, Lemma 6]. ■

In [Gre98a] a space $\mathcal{S}_f G$ is constructed, as a set it consists of the closed subgroups of G , but it does not have the topology induced from the Hausdorff metric. We can relate $\mathcal{S}_f G$ to the space $\mathcal{F}G$ via the equivalence relation \sim on $\mathcal{S}_f G$ which is generated by the following: two subgroups $H \trianglelefteq H'$ are related by \sim if the quotient H'/H is a torus. Then $(\mathcal{S}_f G / \sim) \cong \mathcal{F}G$ and we use tom Dieck's isomorphism to see that an idempotent of $A(G) \otimes \mathbb{Q}$ corresponds to an open and closed G -invariant subspace of $\mathcal{S}_f G$ that is a union of \sim classes.

Definition 1.3.12 Let $a \in A(G) \otimes \mathbb{Q}$, then the **support** of a is the set of $H \leq G$ such that $a(H) \in \mathbb{Q}$ is non-zero (considering a as a continuous map $(\mathcal{S}_f G / \sim)/G \rightarrow \mathbb{Q}$).

Definition 1.3.13 Working in either of $G\mathcal{I}\mathcal{S}$ or $G\mathcal{M}$, a map $f: X \rightarrow Y$ is called an **h -cofibration** if it satisfies the **G -homotopy extension property** defined as follows: whenever there is a pair of maps of G -spectra $F: X \wedge I_+ \rightarrow Z$ and $g: Y \rightarrow Z$ such that $F \circ i_0 = g \circ f$ there exists a map $F': Y \wedge I_+ \rightarrow Z$ making the diagram below commute.

$$\begin{array}{ccccc}
 X & \xrightarrow{i_0} & X \wedge I_+ & & \\
 f \downarrow & & \downarrow F & \searrow f \wedge \text{Id} & \\
 Y & \xrightarrow{g} & Z & & \\
 & \searrow i_0 & & \swarrow F' & \\
 & & Y \wedge I_+ & &
 \end{array}$$

We have taken this definition from [MMSS01, Section 5], as stated in that section there is a universal test case. Let $Z = Mf$ (the mapping cylinder, see Definition 1.4.2), and let g and F be the evident maps. If a suitable F' exists in this case, then f is an h -cofibration. The shorthand notation is $X \xrightarrow{+} Y$.

Lemma 1.3.14 *A map $f: X \rightarrow Y$ is an h -cofibration if and only if it has the left lifting property with respect to the class of maps $ev_0: F(I_+, B) \rightarrow B$. That is, $f: X \rightarrow Y$ is an h -cofibration if and only if every commutative diagram of the form below has a lift.*

$$\begin{array}{ccc}
 X & \xrightarrow{g} & F(I_+, B) \\
 f \downarrow & \nearrow \text{dotted} & \downarrow ev_0 \\
 Y & \xrightarrow{h} & B
 \end{array}$$

Proof It is easy to show that this is an equivalent condition to the definition, perhaps the only point worth noting is that if $h: X \wedge I_+ \rightarrow B$ is the adjoint map to $\tilde{h}: X \rightarrow F(I_+, B)$ then $h \circ i_0 = ev_0 \circ \tilde{h}$. ■

From this description it is clear that pushouts, colimits, retracts and compositions of h -cofibrations are also h -cofibrations. There are some standard maps which are always used in homotopy theory, we have $i_0: S^0 \rightarrow I_+$, it takes the non-basepoint point of S^0 to $0 \in I_+$. Equally there is $i_1: S^0 \rightarrow I_+$ which sends that point to 1. We also need $j: S^0 \rightarrow I$ which includes S^0 as the endpoints into I .

Lemma 1.3.15 *The maps $i_0, i_1: S^0 \rightarrow I_+$ and $j: S^0 \rightarrow I$ are cofibrations of G -spaces.*

Proof The map i_0 is a generating acyclic cofibration of the model category of G -spaces, hence i_1 is a cofibration. Label the points of S_+^0 as 0, 1 and + (the basepoint) and similarly for the endpoints and basepoint of I_+ . We can express S^0 and I as quotients of the spaces S_+^0 and I_+ by the pushout diagrams below.

$$\begin{array}{ccc} \{0\}_+ & \xrightarrow{\quad} & S_+^0 \\ \downarrow & \lrcorner & \downarrow a \\ * & \xrightarrow{\quad} & S^0 \end{array} \qquad \begin{array}{ccc} \{0\}_+ & \xrightarrow{\quad} & I_+ \\ \downarrow & \lrcorner & \downarrow b \\ * & \xrightarrow{\quad} & I \end{array}$$

Now we take a test diagram, with f an acyclic fibration of G -spaces

$$\begin{array}{ccccc} S_+^0 & \xrightarrow{a} & S^0 & \xrightarrow{\alpha} & A \\ \downarrow i & & \downarrow j & \sim \downarrow f & \\ I_+ & \xrightarrow{b} & I & \xrightarrow{\beta} & B \end{array}$$

this diagram gives a lift $g: I_+ \rightarrow A$ and since $g(+) = g(0)$ this passes to a map $h: I \rightarrow A$ and this provides the requisite lift to show that j is a cofibration of G -spaces. ■

Lemma 1.3.16 *If $f: X \rightarrow Y$ is a cofibration of G -spaces and A is a G -spectrum (in $G\mathcal{S}$ or GM), then $\text{Id} \wedge f: A \wedge X \rightarrow A \wedge Y$ is an h -cofibration. Smashing with A preserves h -cofibrations of G -spectra.*

Proof The first statement follows from standard adjunctions relating smashing spectra with G -spaces to the space of maps between two G -spectra. Looking at the universal test case it is easy to see that smashing with a spectrum preserves h -cofibrations (for orthogonal spectra this statement is [MM02, Chapter III, Lemma 7.1]). ■

We now give the analogue of [MM02, Chapter III, Lemma 2.5] for GM .

Lemma 1.3.17 *A cofibration of GM is an h -cofibration*

Proof A generating cofibration has the form $E \rightarrow CE = E \wedge I$ for E a generalised sphere S -module. These are all h -cofibrations of G -spectra since the maps $S^0 \rightarrow I$ and $S^0 \rightarrow I_+$ are cofibrations of G -spaces. Now recall that h -cofibrations are preserved by forming relative cell complexes and retracts to complete the proof. ■

Corollary 1.3.18 *For a G -spectrum X (in $G\mathcal{S}$ or GM), the maps $i_0, i_1: X \rightarrow X \wedge I_+$ and $j: X \rightarrow CX$ are h -cofibrations. In addition, if X is cofibrant, then these maps are cofibrations.*

Proof The first statement follows from Lemma 1.3.16 and the second from the fact that $\wedge: G\mathcal{J}\mathcal{S} \times G\mathcal{T}_* \rightarrow G\mathcal{J}\mathcal{S}$ and $\wedge: G\mathcal{M} \times G\mathcal{T}_* \rightarrow G\mathcal{M}$ are Quillen bifunctors. We note that the operations above are Quillen bifunctors because $G\mathcal{J}\mathcal{S}$ and $G\mathcal{M}$ are G -topological ([MM02, Chapter III, Definition 1.14]), in fact these conditions are equivalent. ■

Lemma 1.3.19 *Every object of $G\mathcal{M}$ is fibrant.*

Proof We must check that following diagram has a lift, for E a generalised sphere spectrum.

$$\begin{array}{ccc} CE & \xrightarrow{f} & X \\ \downarrow i_0 \sim & & \downarrow \\ CE \wedge I_+ & \longrightarrow & * \end{array}$$

There is a retraction map $r: CE \wedge I_+ \rightarrow CE$ such that $r \circ i_0 = 1$ and we take the lift to be $f \circ r$. ■

We can give the analogues of [MM02, Chapter III, Propositions 7.3 and 7.4] for $G\mathcal{M}$.

Lemma 1.3.20 *Let X be a generalised cofibrant spectrum in $G\mathcal{M}$, then the functor $X \wedge -$ preserves weak equivalences. For any acyclic cofibration f and any spectrum Z , the map $f \wedge \text{Id}_Z$ is an h -cofibration and a π_* -isomorphism. Furthermore pushouts and sequential colimits of such maps are h -cofibrations and π_* -isomorphisms.*

Proof The spectrum X is a retract of a generalised cell complex Y . That is: we have the following diagram $X \rightarrow Y \rightarrow X$ with composite map the identity. It suffices to prove this result for generalised cell complexes by the following argument. Assume the result holds for Y and take $f: A \rightarrow B$ a weak equivalence, then we have the diagram:

$$\begin{array}{ccccc} X \wedge A & \longrightarrow & Y \wedge A & \longrightarrow & X \wedge A \\ \downarrow & & \downarrow \sim & & \downarrow \\ X \wedge B & \longrightarrow & Y \wedge B & \longrightarrow & X \wedge B \end{array}$$

hence $X \wedge A \rightarrow X \wedge B$ is a retract of a weak equivalence and thus a weak equivalence. The arguments in [MM02, Chapter III, Proposition 7.3] show that we can reduce this result to proving that if C is a spectrum with trivial homotopy groups, then $C \wedge E$ has trivial homotopy groups for E a generalised sphere S -module. Fix n and a G -indexing space V and consider the generalised sphere S -module $E = S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^\infty(G/H_+ \wedge S^n)$. Since E is strongly dualisable ([LMSM86, Chapter III] or [May96, Chapter XVI, Section 7]) we see (by the conditions on C)

$$\pi_n^K(C \wedge E) = [S^n, C \wedge E]^K \cong [S^n \wedge F(E, S), C]^K = 0.$$

The rest of the lemma follows by the same proof as given for [MMSS01, Proposition 12.5]. For the last statement we use [MM02, Chapter IV, Remark 2.8]. ■

Remark 1.3.21 As stated above $G\mathcal{J}\mathcal{S}$ and GM are Quillen equivalent. In fact [MM02] shows much more, it proves that every construction in GM is the same (up to homotopy) to the corresponding construction in G -equivariant orthogonal spectra. Such constructions include fixed and orbit spectra, geometric fixed point spectra, change of groups and change of universe. So if one were to define an ‘equivariant stable homotopy theory’, then we would have a theorem - both G equivariant S -modules and G -equivariant orthogonal spectra model the same ‘equivariant stable homotopy theory’. When G is the trivial group the category of G -equivariant orthogonal spectra is a model for the stable homotopy category.

1.4 Homotopy Colimits

When one has to construct an object of a model category as a colimit from homotopy level information there will often be choices made in the construction. The correct gadget to organise these choices is a homotopy colimit. Given diagrams C and D in our model category, a map of diagrams $f: C \rightarrow D$ will induce a map $\text{Colim}(f): \text{Colim}(C) \rightarrow \text{Colim}(D)$. If f is an object-wise weak equivalence the induced map will not necessarily be a weak equivalence. A good source for a modern description of these homotopy colimits is [DHKS04]. We will need to use homotopy pushouts and homotopy sequential colimits. All of our categories are cofibrantly generated, thus we can use functorial factorisation in these categories to give some functoriality in our definition of homotopy colimits. Most of the constructions and results of this section hold for quite general model categories. Our main assumptions are that the model category is proper and topological (see [Hov99, Definition 4.2.18]). We have taken the following from [DS95, section 10].

Definition 1.4.1 Consider a diagram $B \leftarrow A \rightarrow C$, let $\widehat{c}A \rightarrow A$ be the cofibrant replacement of A . Factor the maps $\widehat{c}A \rightarrow B$ and $\widehat{c}A \rightarrow C$ into cofibrations followed by acyclic fibrations.

$$\begin{array}{ccccc} B' & \xleftarrow{\quad} & \widehat{c}A & \xrightarrow{\quad} & C' \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ B & \xleftarrow{\quad} & A & \xrightarrow{\quad} & C \end{array}$$

The colimit of $B' \leftarrow \widehat{c}A \rightarrow C'$ is the **homotopy pushout** of $B \leftarrow A \rightarrow C$, written $\text{HoColim}(B \leftarrow A \rightarrow C)$.

This construction has the property that a map of pushout diagrams that is an object-wise weak equivalence induces a weak equivalence between the homotopy pushouts, this is verified in [DS95, section 10]. The reader can also use the definition of a homotopy pushout in [Hir03, 13.5] which has the desired property since our categories are proper (hence left proper). In that section they also prove that there is a weak equivalence between this definition and the construction above.

Definition 1.4.2 Take a map $f: X \rightarrow Y$, then Y/X is the colimit of the diagram $* \leftarrow X \rightarrow Y$. The **mapping cylinder** of f , M_f , is the pushout of the diagram

$X \wedge I_+ \leftarrow X \rightarrow Y$ using the map $i_0: X \rightarrow X \wedge I_+$. The **mapping cone**, Cf , or **cofibre** of f is the colimit of the diagram $CX \leftarrow X \rightarrow Y$ (see the diagram below). The **homotopy cofibre** of f is the homotopy pushout of the diagram $* \leftarrow X \rightarrow Y$.

$$\begin{array}{ccccc}
 & & X & \xrightarrow{f} & Y \\
 & & \downarrow i_0 & \lrcorner & \downarrow \\
 X & \xrightarrow{i_1} & X \wedge I_+ & \longrightarrow & M_f \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 * & \longrightarrow & CX & \longrightarrow & Cf
 \end{array}$$

In the diagram above the maps i_0 and i_1 are the inclusions of X into the 0 and 1 ends of $X \wedge I_+$ and the reader should recall that the map $X \rightarrow CX$ is an h -cofibration. The mapping cone construction has required property that a commutative square as below, such that the vertical maps are weak equivalences, induces a weak equivalence of the cofibres $Cf \rightarrow Cg$.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow \sim & & \downarrow \sim \\
 A & \xrightarrow{g} & B
 \end{array}$$

This is proved by repeated application of [MM02, Chapter III, Theorem 3.5] (or [MM02, Chapter IV, Remark 2.8] for GM) which state that given a diagram as below,

$$\begin{array}{ccccc}
 X & \longleftarrow & Y & \longrightarrow & Z \\
 \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
 X' & \longleftarrow & Y' & \longrightarrow & Z'
 \end{array}$$

the induced map of pushouts is a weak equivalence. The dual construction to Cf gives Ff , the **fibre** of a map, defined in terms of the pullback of f over the map $F(I, Y) \rightarrow Y$. Note that if X and Y are cofibrant then Cf is cofibrant by Corollary 1.3.18 and the fact that cofibrations are preserved by pushouts (using the diagram $CX \leftarrow X \rightarrow Y$). The dual to Y/X is $f^{-1}*$, the pre-image of the basepoint of Y . For the sake of completeness and as a useful exercise in the above definitions we record the result below which can be thought of as saying that the mapping cone is a construction of the homotopy cofibre.

Lemma 1.4.3 *The mapping cone of a map $f: X \rightarrow Y$ is weakly equivalent to the homotopy cofibre of f .*

Proof We begin by replacing X by a cofibrant object, $\widehat{c}X \rightarrow X$ then we factorise the maps $\widehat{c}X \rightarrow *$ and $\widehat{c}X \rightarrow Y$ into cofibrations followed by acyclic fibrations,

$$\begin{array}{ccccc}
 Z' & \longleftarrow & \widehat{c}X & \longrightarrow & Y' \\
 \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
 * & \longleftarrow & X & \longrightarrow & Y
 \end{array}$$

We now draw a comparison diagram

$$\begin{array}{ccccc}
 Z' & \xleftarrow{\quad} & \widehat{C}X & \xrightarrow{f'} & Y' \\
 0 \downarrow \sim & & i_1 \downarrow \sim & & i_1 \downarrow \sim \\
 C\widehat{C}X & \xleftarrow{q} & \widehat{C}X \wedge I_+ & \xrightarrow{f' \wedge 1} & Y \wedge I_+ \\
 \uparrow \sim & & i_0 \uparrow \sim & & i_0 \uparrow \sim \\
 C\widehat{C}X & \xleftarrow{\quad} & +\widehat{C}X & \xrightarrow{f'} & Y'
 \end{array}$$

The map labelled 0 is the map which sends Z' to the basepoint of $C\widehat{C}X$. The maps f' and $f' \wedge 1$ are cofibrations, hence h -cofibrations and all vertical maps are weak equivalences. Thus, the pushouts of the horizontal diagrams are weakly equivalent. The final comparison is

$$\begin{array}{ccccc}
 C\widehat{C}X & \xleftarrow{\quad} & +\widehat{C}X & \xrightarrow{\quad} & Y' \\
 \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
 CX & \xleftarrow{\quad} & +X & \xrightarrow{\quad} & Y
 \end{array}$$

and this induces a weak equivalence of the pushouts by the standard argument. \blacksquare

Lemma 1.4.4 *For any two homotopic maps $f_0, f_1: X \rightarrow Y$ there is a chain of weak equivalences between $\text{cofibre}(f_0)$ and $\text{cofibre}(f_1)$.*

Proof Use the commutative diagram below.

$$\begin{array}{ccccc}
 CX & \xleftarrow{\quad} & +X & \xrightarrow{f_0} & Y \\
 i_0 \downarrow \sim & & i_0 \downarrow \sim & & = \downarrow \sim \\
 CX \wedge I_+ & \xleftarrow{\quad} & +X \wedge I_+ & \xrightarrow{F} & Y \\
 i_1 \uparrow \sim & & i_1 \uparrow \sim & & = \uparrow \sim \\
 CX & \xleftarrow{\quad} & +X & \xrightarrow{f_1} & Y
 \end{array}$$

The maps i_0 and i_1 are (level equivalences and hence) weak equivalences and $X \rightarrow CX$ is an h -cofibration. Hence we apply [MM02, Chapter III, Theorem 3.5] and see that $\text{cofibre}(f_0)$, $\text{cofibre}(F)$ and $\text{cofibre}(f_1)$ are all weakly equivalent. \blacksquare

Lemma 1.4.5 *A homotopy commuting square*

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 p \downarrow & & \downarrow q \\
 C & \xrightarrow{g} & D
 \end{array}$$

induces a zig-zag of maps between $\text{cofibre}(f)$ and $\text{cofibre}(g)$. Furthermore if p and q are weak equivalences then the zig-zag consists of weak equivalences.

Proof We know that $q \circ f \simeq g \circ p$ and thus by Lemma 1.4.4 we know that these maps have weakly equivalent cofibres with a zig-zag of comparisons $\text{cofibre}(q \circ f) \rightarrow \text{cofibre}(F) \leftarrow \text{cofibre}(g \circ p)$. All we need now is the pair of diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow = & & \downarrow q \\ A & \xrightarrow{q \circ f} & D \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g \circ p} & D \\ \downarrow p & & \downarrow = \\ C & \xrightarrow{g} & D \end{array}$$

which will give us the comparison maps $\text{cofibre}(f) \rightarrow \text{cofibre}(q \circ f)$ and $\text{cofibre}(g \circ p) \rightarrow \text{cofibre}(g)$. These are weak equivalences when p and q are. ■

Definition 1.4.6 For a sequence of maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \longrightarrow \dots$$

we define the **homotopy sequential colimit** or **telescope** to be the colimit of the following diagram.

$$\begin{array}{ccccccc} X_0 & \xrightarrow{g_0} & X'_1 & \xrightarrow{g_1} & X'_2 & \xrightarrow{g_2} & X'_3 & \xrightarrow{g_3} & X'_4 & \xrightarrow{g_4} & \dots \\ \downarrow h_0 & & \sim \downarrow h_1 & & \sim \downarrow h_2 & & \sim \downarrow h_3 & & \sim \downarrow h_4 & & \\ X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & \dots \end{array}$$

Where $h_{i+1} \circ g_i$ is a factorisation of $f_i \circ h_i$ into a cofibration followed by a acyclic fibration. We denote this construction by $\text{HoColim}_i X_i$.

Lemma 1.4.7 For a map of colimit diagrams as below, there is an induced map of homotopy colimits $\text{HoColim}_i X_i \rightarrow \text{HoColim}_i Y_i$.

$$\begin{array}{ccccccc} X_0 & \xrightarrow{g_0} & X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \xrightarrow{g_3} & X_4 & \xrightarrow{g_4} & \dots \\ \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 & & \\ Y_0 & \xrightarrow{f_0} & Y_1 & \xrightarrow{f_1} & Y_2 & \xrightarrow{f_2} & Y_3 & \xrightarrow{f_3} & Y_4 & \xrightarrow{f_4} & \dots \end{array}$$

Proof This follows from the fact that we have assumed functorial factorisation in our definition of a model category. ■

This lemma with the following proposition will imply that if the above map of colimit diagrams is an object-wise weak equivalence, then the induced map of homotopy colimits is a weak equivalence.

Proposition 1.4.8 For a map of colimit diagrams that is an object-wise weak equivalence as below, the induced map of colimits is a weak equivalence.

$$\begin{array}{ccccccc} X_0 & \xrightarrow{g_0} & X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \xrightarrow{g_3} & X_4 & \xrightarrow{g_4} & \dots \\ \sim \downarrow h_0 & & \sim \downarrow h_1 & & \sim \downarrow h_2 & & \sim \downarrow h_3 & & \sim \downarrow h_4 & & \\ Y_0 & \xrightarrow{f_0} & Y_1 & \xrightarrow{f_1} & Y_2 & \xrightarrow{f_2} & Y_3 & \xrightarrow{f_3} & Y_4 & \xrightarrow{f_4} & \dots \end{array}$$

Proof We begin by extending the diagram to include the cofibres of the h_i , we have $l_i: Y_i \rightarrow Ch_i$, which is an h -cofibration and $k_i: Ch_i \rightarrow Ch_{i+1}$. The map k_i is an h -cofibration by inspection and since each Ch_i is acyclic (consider the long exact sequence of a cofibration) each k_i is a weak equivalence.

We now apply [MM02, Chapter III, Theorem 3.5 or Chapter IV, Remark 2.8] to see that $\text{Colim}_i(Ch_i)$ is acyclic. Now we apply the standard yoga of colimits and left adjoints (recall that $CX_i = X_i \wedge I$) to see that $\text{Colim}_i(Ch_i) \cong C\text{Colim}_i(h_i)$. Hence, both are acyclic and we can conclude that $\text{Colim}_i(h_i)$ is a weak equivalence. ■

Now we give a rather specific lemma, analogous to Lemma 1.4.5 which allows us to take homotopy level information and use it to create a zig-zag of maps between homotopy colimits. The constructions in this result are standard. We use this lemma in the proof of Theorem 3.2.4.

Lemma 1.4.9 *Consider two sequential colimit diagrams $g_i: X_i \rightarrow X_{i+1}$ and $f_i: Y_i \rightarrow Y_{i+1}$, with each f_i an h -cofibration ($i \geq 0$). Assume that there is a collection of weak equivalences $h_i: X_i \rightarrow Y_i$, with homotopies $F_i: X_i \wedge I \rightarrow Y_i$ such that $F_i \circ i_0 = h_{i+1} \circ g_i$ and $F_i \circ i_1 = f_i \circ h_i$. Then $\text{HoColim}_i X_i$ and $\text{Colim}_i Y_i$ are weakly equivalent.*

Proof We start by drawing the information above as a diagram. We have only assumed that $g_i \circ h_i$ and $h_{i+1} \circ f_i$ are homotopic, so we obtain a homotopy commuting diagram as below.

$$\begin{array}{ccccccc} X_0 & \xrightarrow{g_0} & X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \xrightarrow{g_3} & X_4 & \xrightarrow{g_4} & \cdots \\ \downarrow h_0 & F_0 & \downarrow h_1 & F_1 & \downarrow h_2 & F_2 & \downarrow h_3 & F_3 & \downarrow h_4 & F_4 & \\ Y_0 & \xrightarrow{f_0} & Y_1 & \xrightarrow{f_1} & Y_2 & \xrightarrow{f_2} & Y_3 & \xrightarrow{f_3} & Y_4 & \xrightarrow{f_4} & \cdots \end{array}$$

We now perform the classical construction of the homotopy colimit to replace $g_i: X_i \rightarrow X_{i+1}$ with a sequence of h -cofibrations $k_i: Z_i \rightarrow Z_{i+1}$ with weak equivalences $a_i: X_i \rightarrow Z_i$ and $r_i: Z_i \rightarrow X_i$ such that $r_i \circ a_i = \text{Id}_{X_i}$ and $r_{i+1} \circ k_i = g_i \circ r_i$. Let $Z_0 = X_0$ and $a_0 = r_0 = \text{Id}_{X_0}$, assume inductively we have created stage i , we construct stage $i+1$ of Z_i as a pushout below.

$$\begin{array}{ccc} X_i & \xrightarrow{g_i} & X_{i+1} \\ \downarrow i_0 \circ a_i & \sim & \downarrow a_{i+1} \\ Z_i \wedge I_+ & \xrightarrow{b_i} & Z_{i+1} \end{array}$$

The map $a_{i+1}: X_{i+1} \rightarrow Z_{i+1}$, is an h -cofibration and a weak equivalence by [MM02, Chapter IV, Theorem 3.5]. We define $k_i: Z_i \rightarrow Z_{i+1}$ as the composite $b_i \circ i_1$, which is an h -cofibration. The map r_{i+1} is induced by $\text{Id}_{X_{i+1}}$ and the map $Z_i \wedge I_+ \rightarrow X_{i+1}$ given by retracting to Z_i , then applying $k_i \circ r_i$. Since a_{i+1} is a weak equivalence so is r_i . The collection $r_i: Z_i \rightarrow X_{i+1}$ gives a map of colimit diagrams. Thus we have induced weak equivalences $\text{HoColim}_i Z_i \rightarrow \text{Colim}_i Z_i$ and $\text{HoColim}_i Z_i \rightarrow \text{HoColim}_i X_i$.

Now we compare the Z_i to the Y_i , this is slightly more complicated. We will construct weak equivalences $m_i: Z_i \rightarrow Y_i$ and homotopies $G_i: Z_i \wedge I_+ \rightarrow Y_{i+1}$ such that $m_i \circ a_i = h_i$, and $m_{i+1} \circ b_i = G_i$. Since $Z_0 = X_0$, we can take $m_0 = h_0$. We construct G_0 using

the fact that a_0 is an h -cofibration.

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{i_1} & X_0 \wedge I_+ & & \\
 a_0 \downarrow & & \downarrow F_0 & \searrow a_0 \wedge \text{Id} & \\
 Z_0 & \xrightarrow{f_0 \circ m_0} & Y_1 & \xleftarrow{G_0} & \\
 & \searrow i_1 & & & \downarrow \\
 & & & & Z_0 \wedge I_+
 \end{array}$$

From the definition of Z_1 the maps $h_1: X_1 \rightarrow Y_1$ and $G_0: Z_0 \wedge I_+ \rightarrow Y_1$ induce $m_1: Z_1 \rightarrow Y_1$. We now inductively make m_{i+1} from G_i and G_{i+1} from m_{i+1} . The m_i induce a map of colimits $\text{Colim}_i Z_i \rightarrow \text{Colim}_i Y_i$. Since $m_i \circ a_i = h_i$, m_i is a weak equivalence for all $i \geq 1$. Hence the induced map of colimits is a weak equivalence. ■

Proposition 1.4.10 *If X_0 is cofibrant then $\text{HoColim}_i X_i$ is cofibrant. If either of X_0 or Y is cofibrant then there is a weak equivalence*

$$\text{HoColim}_i(Y \wedge X_i) \rightarrow Y \wedge \text{HoColim}_i(X_i).$$

Proof This follows by standard manipulations of the definitions and the fact that smashing with a cofibrant object preserves weak equivalences. ■

1.5 Rational Sphere Spectra

We construct a ‘rational sphere spectrum’ which we will use in Section 2.2 to construct model categories of rational spectra. We will construct \mathbb{Q} as a group and translate this into spectra. We will do this for both GM and $G\mathcal{I}\mathcal{S}$. Later we will specialise to GM and make a rational sphere spectrum that is a commutative ring spectrum. In order to make the following clear we use \mathbb{S} for the unit of the smash product of $G\mathcal{I}\mathcal{S}$ and S the unit for GM , we work in $G\mathcal{I}\mathcal{S}$ to begin with.

Take a free resolution of \mathbb{Q} as an abelian group, $0 \rightarrow R \xrightarrow{f} F \rightarrow \mathbb{Q} \rightarrow 0$. Since a free abelian group is a direct sum of copies of \mathbb{Z} we can rewrite this short exact sequence as $0 \rightarrow \bigoplus_i \mathbb{Z} \xrightarrow{f} \bigoplus_j \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow 0$. Since \mathbb{Q} is flat, the sequence $0 \rightarrow \bigoplus_i M \xrightarrow{f} \bigoplus_j M \rightarrow \mathbb{Q} \otimes M \rightarrow 0$ is exact for any abelian group M . Hence for each subgroup H of G , we have an injective map (which we also denote as f) $\bigoplus_i A(H) \xrightarrow{f} \bigoplus_j A(H)$ and $\bigoplus_j A(H) / \bigoplus_i A(H) \cong A(H) \otimes \mathbb{Q}$.

Lemma 1.5.1 *For H , a subgroup of G ,*

$$[\bigvee_i \mathbb{S}, \bigvee_j \mathbb{S}]^H \cong \prod_i \bigoplus_j [\mathbb{S}, \mathbb{S}]^H \cong \prod_i \bigoplus_j A(H) \cong \text{Hom}_{A(H)} \left(\bigoplus_i A(H), \bigoplus_j A(H) \right).$$

Proof Since maps out a coproduct is a product of maps we have an isomorphism

$$[\bigvee_i \mathbb{S}, \bigvee_j \mathbb{S}]^H \cong \prod_i [\mathbb{S}, \bigvee_j \mathbb{S}]^H.$$

Now we apply the fact that \mathbb{S} is H -compact and known isomorphisms to obtain the result. \blacksquare

Thus we can choose $g: \widehat{f}\bigvee_i \mathbb{S} \rightarrow \widehat{f}\bigvee_j \mathbb{S}$, a representative for the homotopy class corresponding to f .

Definition 1.5.2 For the map g as constructed above, the cofibre of g is the **rational sphere spectrum** and we have a cofibre sequence

$$\widehat{f}\bigvee_i \mathbb{S} \xrightarrow{g} \widehat{f}\bigvee_j \mathbb{S} \longrightarrow S^0\mathbb{Q}.$$

A different choice of representative for the homotopy class $[g]$ will induce a weak equivalence between the cofibres, and hence (up to weak equivalence) $S^0\mathbb{Q}$ is independent of this choice of representative. Returning to GM we can perform the analogous construction: $\bigvee_i \widehat{c}S \rightarrow \bigvee_j \widehat{c}S \rightarrow S^0_{\mathcal{M}}\mathbb{Q}$, where we need to take cofibrant replacements (since S is not cofibrant in GM). For $G\mathcal{J}\mathcal{S}_+$ we construct $\widehat{f}\bigvee_i \widehat{c}S \rightarrow \widehat{f}\bigvee_j \widehat{c}S \rightarrow S^0\mathbb{Q}_+$. Note that there is an inclusion $\alpha: \mathbb{S} \rightarrow \widehat{f}\bigvee_j \mathbb{S}$ which sends \mathbb{S} to the term of $\bigvee_j \mathbb{S}$ corresponding to $1 \in \mathbb{Q}$. We have similar maps for GM and $G\mathcal{J}\mathcal{S}_+$ using $\widehat{c}S \rightarrow \bigvee_j \widehat{c}S$ and $\widehat{c}S \rightarrow \widehat{f}\bigvee_j \widehat{c}S$.

Proposition 1.5.3 *The rational sphere spectra $S^0\mathbb{Q}$, $S^0_{\mathcal{M}}\mathbb{Q}$ and $S^0\mathbb{Q}_+$ are cofibrant.*

Proof We give the proof for $G\mathcal{J}\mathcal{S}$, the other cases follow by the same argument. The diagram below gives the definition of the cofibre of g .

$$\begin{array}{ccc} \widehat{f}\bigvee_i \mathbb{S} & \xrightarrow{g} & \widehat{f}\bigvee_j \mathbb{S} \\ \downarrow j & \lrcorner & \downarrow \\ C\widehat{f}\bigvee_i \mathbb{S} & \longrightarrow & Cg \end{array}$$

Since a wedge of cofibrant objects is again cofibrant, $\bigvee_i \mathbb{S}$ is cofibrant, hence so is $\widehat{f}\bigvee_i \mathbb{S}$. Corollary 1.3.18 tells us that j is a cofibration and cofibrations are preserved by pushouts, so $\bigvee_j \mathbb{S} \rightarrow Cg$ is a cofibration from a cofibrant object to Cg . \blacksquare

Proposition 1.5.4 *Let X be an orthogonal G -spectrum, then for any subgroup H of G there is a natural isomorphism $\pi_*^H(X \wedge S^0\mathbb{Q}) \cong \pi_*^H(X) \otimes \mathbb{Q}$. Furthermore, the map $(\text{Id}_X \wedge \alpha)_*: \pi_*^H(X) \rightarrow \pi_*^H(S^0\mathbb{Q} \wedge X)$ acts as $x \rightarrow x \otimes 1$. The corresponding statement holds in GM : there is an isomorphism $\pi_*^H(X \wedge S^0\mathbb{Q}) \cong \pi_*^H(X) \otimes \mathbb{Q}$. The map $\pi_*^H(\widehat{c}S \wedge X) \rightarrow \pi_*^H(S^0\mathbb{Q} \wedge X)$ acts as $x \rightarrow x \otimes 1$.*

Proof Using the cofibre sequence which defines $S^0\mathbb{Q}$ we have the following collection of isomorphic long exact sequences of homotopy groups

$$\begin{array}{ccccccc} \dots & \longrightarrow & \pi_n^H(X \wedge \widehat{f}\bigvee_i \mathbb{S}) & \xrightarrow{(\text{Id} \wedge g)^*} & \pi_n^H(X \wedge \widehat{f}\bigvee_j \mathbb{S}) & \longrightarrow & \pi_n^H(X \wedge S^0\mathbb{Q}) \longrightarrow \dots \\ \dots & \longrightarrow & \pi_n^H(\bigvee_i X) & \xrightarrow{(\text{Id} \wedge g)^*} & \pi_n^H(\bigvee_j X) & \longrightarrow & \pi_n^H(X \wedge S^0\mathbb{Q}) \longrightarrow \dots \\ \dots & \longrightarrow & \bigoplus_i \pi_n^H(X) & \xrightarrow{g \otimes \text{Id}} & \bigoplus_j \pi_n^H(X) & \longrightarrow & \pi_n^H(X \wedge S^0\mathbb{Q}) \longrightarrow \dots \\ \dots & \longrightarrow & \bigoplus_i \mathbb{Z} \otimes \pi_n^H(X) & \xrightarrow{g \otimes \text{Id}} & \bigoplus_j \mathbb{Z} \otimes \pi_n^H(X) & \longrightarrow & \pi_n^H(X \wedge S^0\mathbb{Q}) \longrightarrow \dots \end{array}$$

Since the map $g \otimes \text{Id}: (\bigoplus_i \mathbb{Z}) \otimes \pi_n^H(X) \rightarrow (\bigoplus_j \mathbb{Z}) \otimes \pi_n^H(X)$ is injective for all n , this long exact sequence splits into short exact sequences and we conclude that $\pi_*^H(X \wedge S^0 \mathbb{Q}) \cong \pi_*^H(X) \otimes \mathbb{Q}$. The calculation of $(\text{Id}_X \wedge \alpha)_*$ follows immediately, as do the statements regarding $G\mathcal{M}$. ■

Lemma 1.5.5 *The positive rational sphere spectrum $S^0 \mathbb{Q}_+$ is weakly equivalent to the rational sphere spectrum, $S^0 \mathbb{Q}$. The rational sphere spectra $\mathbb{N}S^0 \mathbb{Q}_+$ and $S_{\mathcal{M}}^0 \mathbb{Q}$ are weakly equivalent.*

Proof The first statement follows from the fact that the positive cofibrant replacement of \mathbb{S} is weakly equivalent to \mathbb{S} . For the second statement, let c and f denote fibrant replacement in $G\mathcal{I}\mathcal{S}_+$ and \hat{c} be cofibrant replacement in $G\mathcal{M}$. Since \mathbb{N} is a strong monoidal left adjoint, we know that the map $\mathbb{N}c\mathbb{S} \rightarrow \mathbb{N}\mathbb{S} \cong S$ is a weak equivalence. Hence we can choose (via lifting properties) a weak equivalence $\mathbb{N}c\mathbb{S} \rightarrow \hat{c}S$. We can then make the following diagram, which commutes up to homotopy. Each of the horizontal maps is some choice of a representative for g .

$$\begin{array}{ccc}
 \mathbb{N}fV_j c\mathbb{S} & \longrightarrow & \mathbb{N}fV_i c\mathbb{S} \\
 \uparrow \sim & & \uparrow \sim \\
 \mathbb{N}V_j c\mathbb{S} & \longrightarrow & \mathbb{N}V_i c\mathbb{S} \\
 \downarrow \sim & & \downarrow \sim \\
 V_j \hat{c}S & \longrightarrow & V_i \hat{c}S
 \end{array}$$

The vertical maps are weak equivalences, hence the cofibres of the horizontal maps are weakly equivalent. ■

Later it will be important to have a commutative ring spectrum that we can call a rational sphere, we give a method to construct such a object. We work exclusively in $G\mathcal{M}$ in the following. We note here that the classical concept of cell spectra can be reworded to: X is a cell spectrum if the map $*$ \rightarrow X is in I -cell. We may as well have made $S_{\mathcal{M}}^0 \mathbb{Q}$ in $G\mathcal{M}$ from copies of a cellular replacement of the sphere spectrum (see [EKMM97, Chapter III, Theorem 2.10]). If we do so, then we have the following lemma.

Lemma 1.5.6 *The rational sphere spectrum $S_{\mathcal{M}}^0 \mathbb{Q}$ is a cell S -module.*

Proof Recall the following from [LMSM86, Chapter I, Lemma 5.7]: a wedge of cell S -modules is a cell S -module, for a cell S -module X the canonical map $X \rightarrow CX$ is an inclusion of a cell subcomplex and the pushout of a cellular map of cell S -modules along an inclusion of a cell subcomplex is a cell S -module. Now we use [LMSM86, Chapter I, Theorem 5.8] (Cellular Approximation) to ensure that the map $f: V_i \Sigma^\infty S^0 \rightarrow V_j \Sigma^\infty S^0$ is cellular and the result follows since $\Sigma^\infty S^0$ is a cell S -module. ■

To obtain our commutative ring spectrum we use [EKMM97, Chapter VIII, Theorem 2.2], we give the statement that we will need below. Here we assume that E is a

cell spectrum (hence cofibrant). We need to use the language of E -equivalences and E -localisation from 2.1.

Theorem 1.5.7 *For a cell commutative R -algebra A , the localisation $\lambda: A \rightarrow A_E$ can be constructed as the inclusion of a subcomplex in a cell commutative R -algebra A_E . In particular $A \rightarrow A_E$ is an E -equivalence and a cofibration of commutative ring spectra for any cell commutative R -algebra A .*

Proof This result goes through with the same proof as in [EKMM97]. ■

Definition 1.5.8 Let $S_{\mathbb{Q}}$ be the commutative ring spectrum constructed as the $S_{\mathcal{M}}^0\mathbb{Q}$ -localisation of S in $G\mathcal{M}$ with the generalised cellular model structure.

It follows immediately that the unit $\eta: S \rightarrow S_{\mathbb{Q}}$ is an $S_{\mathcal{M}}^0\mathbb{Q}$ -equivalence.

Lemma 1.5.9 *There is a weak equivalence $S_{\mathcal{M}}^0\mathbb{Q} \rightarrow S_{\mathbb{Q}}$.*

Proof Begin with the map $\widehat{c}S \rightarrow \bigvee_j \widehat{c}S$ which maps into the $\widehat{c}S$ factor corresponding to $1 \in \mathbb{Q}$. This is clearly a cofibration, hence we obtain a cofibration $\widehat{c}S \rightarrow S_{\mathcal{M}}^0\mathbb{Q}$ that is an isomorphism of rational homotopy groups and hence by Lemma 2.2.1 is an $S_{\mathcal{M}}^0\mathbb{Q}$ -equivalence. We then have the composite map $\widehat{c}S \rightarrow S \rightarrow S_{\mathbb{Q}}$ which is a $S_{\mathcal{M}}^0\mathbb{Q}$ -equivalence into an $S_{\mathcal{M}}^0\mathbb{Q}$ -local object. Thus we obtain a $S_{\mathcal{M}}^0\mathbb{Q}$ -equivalence $S_{\mathcal{M}}^0\mathbb{Q} \rightarrow S_{\mathbb{Q}}$ (via the lifting properties in $L_{S_{\mathcal{M}}^0\mathbb{Q}}G\mathcal{M}$). By Theorem 2.2.4, $S_{\mathbb{Q}}$ has rational homotopy groups thus our lift must be a π_* -isomorphism. ■

Chapter 2

Localisations of G -Spectra

We will define our category of rational spectra in terms of a Bousfield localisation of $G\mathcal{J}\mathcal{S}$ or GM . We will need other localisations later, so we begin by looking at the general case as considered in [MM02, Chapter IV, Section 6], our contribution will be in proving that most of the good model category properties of G -spectra are preserved by localisation (left properness, the pushout product axiom and the monoid axiom). We then construct our category of rational G -spectra in Section 2.2 and prove that it is independent of the choices in our construction (see Lemma 2.2.1 and Propositions 2.2.6 and 2.2.7). We give a different construction in terms of modules over a ring spectrum in Section 2.3 and show that this construction is Quillen equivalent to the previous one (Theorem 2.3.4).

2.1 The Bousfield Localisations of $G\mathcal{J}\mathcal{S}$ and GM

The results below apply to $G\mathcal{J}\mathcal{S}$, $G\mathcal{J}\mathcal{S}_+$ and GM so we work with notation appropriate to $G\mathcal{J}\mathcal{S}$ and will note when changes are needed for GM or $G\mathcal{J}\mathcal{S}_+$. Aside from the theorem below which allows us to actually construct these localisations, the major results are Theorem 2.1.12 and Proposition 2.1.18. The first allows us to compare localised categories and the second shows that we can localise categories of modules over a ring spectrum. The definition below is [MM02, Chapter IV, Definition 6.2].

Definition 2.1.1 Let E be a cofibrant spectrum or a cofibrant based G -space and let X , Y and Z be orthogonal spectra.

- (i). A map $f: X \rightarrow Y$ is an E -**equivalence** if $\text{Id}_E \wedge f: E \wedge X \rightarrow E \wedge Y$ is a weak equivalence.
- (ii). Z is E -**local** if $f^*: [Y, Z]^G \rightarrow [X, Z]^G$ is an isomorphism for all E -equivalences $f: X \rightarrow Y$.
- (iii). An E -**localisation** of X is an E -equivalence $\lambda: X \rightarrow Y$ from X to an E -local object Y .

(iv). A is E -**acyclic** if the map $* \rightarrow A$ is an E -equivalence.

The following is a standard result, see [Hir03, Theorems 3.2.13 and 3.2.14].

Lemma 2.1.2 *An E -equivalence between E -local objects is a weak equivalence.*

Proof Take $f: A \rightarrow B$, an E -equivalence between E -local objects. The square below commutes and the vertical maps are isomorphisms by the definitions above.

$$\begin{array}{ccc} [B, A]^G & \xrightarrow{f_*} & [B, B]^G \\ f^* \downarrow \cong & & f^* \downarrow \cong \\ [A, A]^G & \xrightarrow{f_*} & [A, B]^G \end{array}$$

There is a unique $[g] \in [B, A]^G$ such that $[g][f] = f^*[g] = [\text{Id}_A]$ and it follows that $[f][g] = f_*[g] = [\text{Id}_B]$. Thus $[f]$ is an isomorphism in the homotopy category and we can conclude that f is a weak equivalence. ■

Theorem 2.1.3 *Let E be a cofibrant spectrum or a cofibrant based G -space. Then $G\mathcal{I}\mathcal{S}$ has an E -model structures whose weak equivalences are the E -equivalences and whose E -cofibrations are the cofibrations of $G\mathcal{I}\mathcal{S}$. The E -fibrant objects are precisely the fibrant E -local objects and E -fibrant approximation constructs a Bousfield localisation $f_X: X \rightarrow \widehat{f}_E X$ of X at E . The notation for E -model structure on the underlying category of $G\mathcal{I}\mathcal{S}$ is $L_E G\mathcal{I}\mathcal{S}$ or $G\mathcal{I}\mathcal{S}_E$.*

Proof This result is [MM02, Chapter IV, Theorem 6.3], the proof of which is an adaptation of the material in [EKMM97, chapter VIII]. ■

These are cofibrantly generated model categories, this is seen by carefully reading the proof of [EKMM97, Chapter VIII, Theorem 1.1]. Let c be a fixed infinite cardinal that is at least the cardinality of $E^*(\mathbb{S})$. Then define \mathcal{T} , a test set for E -fibrations, to consist of all inclusions of cell complexes $X \rightarrow Y$ such that the cardinality of the set of cells of Y is less than or equal to c . Hence the domains of these maps are κ -small where κ is the least cardinal greater than c .

Proposition 2.1.4 *The identity map of $G\mathcal{I}\mathcal{S}$ is the left adjoint of a Quillen pair from $G\mathcal{I}\mathcal{S} \rightarrow L_E G\mathcal{I}\mathcal{S}$.*

Proof The cofibrations are unchanged and a π_* -isomorphism is an E -equivalence since weak equivalences are preserved by smashing with a cofibrant object. ■

Lemma 2.1.5 *A spectrum X is E -local if and only if $[A, X]^G = 0$ for all E -acyclic spectra A .*

Proof We tackle the only if part first. Assume X is E -local, take an E -acyclic spectrum A , then since the map $* \rightarrow A$ is an E -equivalence, $[A, X]^G \cong [*, X]^G = 0$.

For the converse we assume that $[A, X]^G = 0$ for all E -acyclic spectra A . Since this is a homotopy level condition we may as well assume that X is fibrant. Take an E -equivalence $f: Y \rightarrow Z$ then we must prove that $[Z, X]^G \cong [Y, X]^G$. Hence we can

assume that X and Y are cofibrant, thus Cf is cofibrant. By the long exact sequence of homotopy groups of a cofibre, it follows that $Cf \wedge E \cong C(f \wedge \text{Id}_E)$ is acyclic, hence Cf is E -acyclic. We have a fibre sequence $F(Cf, X) \rightarrow F(Z, X) \rightarrow F(Y, X)$, Looking at the long exact sequence of homotopy groups of a fibre, we see that $F(Cf, X)$ is acyclic if and only if $F(f, \text{Id}_X)$ is a weak equivalence. We have an isomorphism $\pi_*^H(F(Cf, X)) \cong [G/H_+ \wedge Cf, X]_*^G$ and since Cf (and hence $G/H_+ \wedge Cf$) is E -acyclic and X is E -local we see that this is zero. The weak equivalence $F(Z, X) \rightarrow F(Y, X)$ gives an isomorphism $[Z, X]^G \cong [Y, X]^G$ as desired. ■

Now we turn our attention to proving some useful results on the E -local model structures. We write \widehat{f}_E for fibrant replacement in the E -local model categories and maps in the E -local homotopy category between spectra X and Y will be written $[X, Y]_E$.

Proposition 2.1.6 *For two cofibrations, $f: U \rightarrow V$ and $g: W \rightarrow X$, the induced map*

$$f \square g: V \wedge W \bigvee_{U \wedge W} U \wedge X \rightarrow V \wedge X$$

is a cofibration which is an E -acyclic cofibration if either f or g is. If X is a cofibrant spectrum then the map $\widehat{cS} \wedge X \rightarrow X$ is a weak equivalence.

Proof Since the cofibrations are unchanged by localisation, we only need to check that the above map is an E -equivalence when f is. The map $f \wedge \text{Id}_E: U \wedge E \rightarrow V \wedge E$ is a π_* -isomorphism and a cofibration. Thus since $(-) \wedge E$ commutes with pushouts the map

$$(V \wedge W \bigvee_{U \wedge W} U \wedge X) \wedge E \rightarrow (V \wedge X) \wedge E$$

is also a π_* -isomorphism and a cofibration. The unit condition is unaffected by localisation, so it holds in the E -local model structure. ■

Thus $G\mathcal{S}\mathcal{S}_E$ is a monoidal model category. The above proof is our standard method for moving results to the E -local model structures for cofibrant E , we use it to prove the following.

Proposition 2.1.7 (i). *A map is an E -equivalence if and only if its suspension is an E -equivalence.*

(ii). *A wedge of E -equivalences is an E -equivalence.*

(iii). *If $i: A \rightarrow X$ is an h -cofibration and an E -equivalence and $f: A \rightarrow Y$ is any map, then the cobase change $j: Y \rightarrow X \vee_A Y$ is an E -equivalence.*

(iv). *If i and i' are h -cofibrations and the vertical arrows are E -equivalences in the diagram below, then the induced map of pushouts is an E -equivalence.*

$$\begin{array}{ccccc} X & \xleftarrow{i} & A & \longrightarrow & Y \\ \downarrow \sim E & & \downarrow \sim E & & \downarrow \sim E \\ X' & \xleftarrow{i'} & A' & \longrightarrow & Y' \end{array}$$

(v). If X is the colimit of a sequence of h -fibrations $X_n \rightarrow X_{n+1}$, each of which is an E -equivalence, then the map from the initial term X_0 into X is an E -equivalence.

Proof This follows from [MM02, Chapter III, Theorem 3.5] for orthogonal spectra and from [MM02, Chapter IV, Remark 2.8] for GM . ■

Proposition 2.1.8 *For a cofibrant orthogonal G -spectrum X , the functor $X \wedge (-)$ preserves E -equivalences.*

Proof The functor $X \wedge (-)$ preserves π_* -isomorphisms hence the result follows from the associativity of the smash product. ■

Proposition 2.1.9 *For $i: A \rightarrow X$ an acyclic E -cofibration and any spectrum Y , the map $i \wedge \text{Id}_Y: A \wedge Y \rightarrow X \wedge Y$ is an E -equivalence and an h -cofibration. Moreover, cobase changes and sequential colimits of such maps are E -equivalences and h -cofibrations.*

Proof Use [MM02, Chapter III, Lemma 7.1] to see that $i \wedge \text{Id}_Y$ is an h -cofibration. By [MM02, Chapter III, Proposition 7.4] $i \wedge \text{Id}_E \wedge \text{Id}_Y$ is an h -cofibration and a π_* -isomorphism, hence $i \wedge \text{Id}_Y$ is an E -equivalence.

We have proved the first statement of the proposition, the second follows from Proposition 2.1.7 and the fact that h -cofibrations are closed under pushouts and sequential colimits. ■

For GM we follow the same proof using Lemma 1.3.20.

Proposition 2.1.10 *The model category $G\mathcal{S}\mathcal{S}_E$ is left proper.*

Proof We show a stronger result: that in the pushout diagram below, with α an h -cofibration the map labelled l is an E -equivalence. Since a cofibration is an h -cofibration this implies left properness.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \sim_E \downarrow & \ulcorner & \downarrow l \\ C & \xrightarrow{\gamma} & D \end{array}$$

The functor $(-) \wedge E$ preserves h -cofibrations, pushouts and takes E -equivalences (\sim_E) to π_* -isomorphisms. We apply this functor to the pushout diagram above and use left properness of $G\mathcal{S}\mathcal{S}$, [MM02, Chapter III, Lemma 4.13], (or [MM02, Chapter IV, Theorem 2.9] for GM) to see that $l \wedge \text{Id}_E$ is a π_* -isomorphism. Thus l is an E -equivalence as desired. ■

See Lemmas 2.2.3 and 3.4.13 for right properness in the cases of most interest to us.

Remark 2.1.11 The spectra G/H_+ where H runs over all subgroups H of G are generators for $G\mathcal{S}\mathcal{S}_E$. This well known fact follows from Lemma 1.3.6.

Theorem 2.1.12 *Take a Quillen adjunction between monoidal model categories with a strong monoidal left adjoint $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$. Let E be cofibrant in \mathcal{C} and assume that the model categories \mathcal{C}_E and \mathcal{D}_{FE} exist. Then (F, G) passes to a strong monoidal Quillen pair $F : \mathcal{C}_E \rightleftarrows \mathcal{D}_{FE} : G$. Furthermore, if (F, G) form a Quillen equivalence, then they pass to a Quillen equivalence of the localised categories.*

Proof Since the cofibrations in \mathcal{C}_E and \mathcal{D}_{FE} are unchanged F preserves cofibrations. Now take an acyclic cofibration in \mathcal{C} of the form $f \wedge \text{Id}_E : X \wedge E \rightarrow Y \wedge E$, applying F and using the strong monoidal condition we have a weak equivalence in \mathcal{D} , $Ff \wedge \text{Id}_{FE} : FX \wedge FE \rightarrow FY \wedge FE$. Hence F takes E -acyclic cofibrations to FE -acyclic cofibrations and we have a Quillen pair.

To prove the second statement we show that F reflects E -equivalences between cofibrant objects and that $F\hat{c}GX \rightarrow X$ is an E -equivalence for all X fibrant in \mathcal{D}_{FE} . These conditions are an equivalent definition of Quillen equivalence by [Hov99, Corollary 1.3.16(b)]. The first follows since strong monoidality allows us to identify $F(f \wedge \text{Id}_E)$ and $Ff \wedge \text{Id}_{FE}$ for a map f in \mathcal{C} and F reflects weak equivalences between cofibrant objects. The second condition is equally simple: we know that an E -fibrant object is fibrant, and that cofibrant replacement is unaffected by Bousfield localisation. Hence $F\hat{c}GX \rightarrow X$ is a weak equivalence and thus an E -equivalence. ■

Corollary 2.1.13 *For E , a positive cofibrant orthogonal G -spectrum, the Quillen equivalence $\mathbb{N} : G\mathcal{J}\mathcal{S}_+ \rightleftarrows G\mathcal{M} : \mathbb{N}^\#$ passes to a Quillen equivalence*

$$\mathbb{N} : L_E G\mathcal{J}\mathcal{S}_+ \rightleftarrows L_{NE} G\mathcal{M} : \mathbb{N}^\#.$$

Remark 2.1.14 If E is positive cofibrant in $G\mathcal{J}\mathcal{S}_+$ then the identity functor is the left adjoint of a Quillen equivalence from $(G\mathcal{J}\mathcal{S}_+)_E$ to $G\mathcal{J}\mathcal{S}_E$. Of course, if E' is a cofibrant spectrum in $G\mathcal{J}\mathcal{S}$ and $\hat{c}_+ E'$ is its positive cofibrant replacement then $G\mathcal{J}\mathcal{S}_{E'} = G\mathcal{J}\mathcal{S}_{\hat{c}_+ E'}$ hence $G\mathcal{J}\mathcal{S}_{E'}$ is Quillen equivalent by $(G\mathcal{J}\mathcal{S}_+)_{\hat{c}_+ E'}$.

Theorem 2.1.12 implies that the forgetful functor ι_H^* and the inflation functor ε_G^* (and their right adjoints $F_H(G_+, -)$ and $(-)^G$) pass to Quillen functors on the E -local categories.

Lemma 2.1.15 *The pair $(G_+ \wedge_H (-), \iota_H^*)$ pass to a Quillen pair between the categories $L_{\iota_H^* E} H\mathcal{J}\mathcal{S}$ and $L_E G\mathcal{J}\mathcal{S}$.*

Proof This does not follow from the above theorem. As always $G_+ \wedge_H (-)$ preserves cofibrations in the E -local model structure. The isomorphism ([MM02, Chapter V, Proposition 2.3]) $(G_+ \wedge_H X) \wedge Y \cong G_+ \wedge_H (X \wedge \iota_H^* Y)$, which is natural in H -spectra X and G -spectra Y , applied to the case $Y = E$ proves that the left adjoint preserves the acyclic cofibrations of the localised category. ■

It is clear that one could attempt to repeat the process of localisation and localise $G\mathcal{J}\mathcal{S}_E$ at F (either a cofibrant spectrum or a cofibrant G -space). The cofibrations would be unchanged, the weak equivalences would be those maps $f : X \rightarrow Y$ such that $f \wedge \text{Id}_F : X \wedge F \rightarrow Y \wedge F$ is an E -equivalence and the fibrations would be given by a lifting property. Initially it appears that one would have to check that this does give

a model category, essentially reproving Theorem 2.1.3 for $L_E G\mathcal{S}\mathcal{S}$, but this is not the case. One simply has to notice that these new weak equivalences are $(F \wedge E)$ -equivalences and that this proposed new model structure on $L_E G\mathcal{S}\mathcal{S}$ coincides with the $(F \wedge E)$ -model structure on $G\mathcal{S}\mathcal{S}$. We have proved the theorem below.

Theorem 2.1.16 *Let F be a cofibrant object of $G\mathcal{S}\mathcal{S}_E$ (or, equally, of $G\mathcal{S}\mathcal{S}$) or a cofibrant based G -space. Then $L_E G\mathcal{S}\mathcal{S}$ has a Bousfield F -model structure with weak equivalences the $(F \wedge E)$ -equivalences and cofibrations the cofibrations of $G\mathcal{S}\mathcal{S}$. The fibrant objects are precisely the fibrant $(F \wedge E)$ -local objects of $G\mathcal{S}\mathcal{S}$ and $(F \wedge E)$ -fibrant approximation constructs a Bousfield localisation $f_X: X \rightarrow \hat{f}_{F \wedge E} X$ of X at $F \wedge E$. The notation for the F -model structure on the underlying category of $G\mathcal{S}\mathcal{S}$ is $L_F L_E G\mathcal{S}\mathcal{S}$. Furthermore we have the following identifications of model categories*

$$L_F L_E G\mathcal{S}\mathcal{S} = L_{F \wedge E} G\mathcal{S}\mathcal{S} = L_{E \wedge F} G\mathcal{S}\mathcal{S} = L_E L_F G\mathcal{S}\mathcal{S}.$$

Combining the above theorem with the various results of this section we have the following summary.

Corollary 2.1.17 *The category $L_F L_E G\mathcal{S}\mathcal{S}$ is a cofibrantly generated, left proper, monoidal model category satisfying the monoid axiom, the spectra $\mathbb{S} \wedge G/H_+$ for H a subgroup of G form a countable set of generators.*

Proposition 2.1.18 *Let R be a commutative ring spectrum and E be a cofibrant G -spectrum or cofibrant G -space, then there is a model structure $L_E(R\text{-mod})$ on R -modules with weak equivalences and fibrations the E -equivalences and E -fibrations of underlying spectra and cofibrations as for $R\text{-mod}$.*

Proof The E -local model structure on G -spectra gives rise to the model category as defined in the theorem since it satisfies the monoid axiom. \blacksquare

Note that $L_E(R\text{-mod})$ is precisely the model structure of $R\text{-mod}$ localised at $E \wedge R$. It is easily seen that the weak equivalences and cofibrations are the same.

2.2 The categories $G\mathcal{S}\mathcal{S}_{\mathbb{Q}}$ and $GM_{\mathbb{Q}}$

Take $E = S^0\mathbb{Q}$, we will call the E -local model structure the rational model structure and write $L_{S^0\mathbb{Q}} G\mathcal{S}\mathcal{S}$ or $G\mathcal{S}\mathcal{S}_{\mathbb{Q}}$. For GM we take $S^0_M\mathbb{Q}$ and write the localised category as $GM_{\mathbb{Q}}$. We will write $G\mathcal{S}\mathcal{S}_{\mathbb{Q}}^+$ for the rationalisation of the positive model structure $G\mathcal{S}\mathcal{S}_+$ (using $S^0\mathbb{Q}_+$). We call E -equivalences **rational equivalences**, or $\pi_*^{\mathbb{Q}}$ -**isomorphisms**. Equally, E -fibrations will be called **rational fibrations** and acyclic E -cofibrations will be called **acyclic rational cofibrations**. The set of rational homotopy classes of maps X to Y will be written $[X, Y]_{\mathbb{Q}}^G$ and we will write $\hat{f}_{\mathbb{Q}}$ for fibrant replacement in the localised category. Note that since the cofibrations agree, the rational acyclic fibrations are the acyclic fibrations of $G\mathcal{S}\mathcal{S}$. Hence factorising a map into a cofibration followed by a rational acyclic fibration is the same operation in both $G\mathcal{S}\mathcal{S}$ and $G\mathcal{S}\mathcal{S}_{\mathbb{Q}}$. We will prove that our rationalised categories $G\mathcal{S}\mathcal{S}_{\mathbb{Q}}$,

$G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+$ and $GM_{\mathbb{Q}}$ are Quillen equivalent (Propositions 2.2.6 and 2.2.7), so that we can switch between these at will. The lemma below shows that our model structure is independent of our choice of rational sphere spectrum.

Lemma 2.2.1 *For a map $g: X \rightarrow Y$ the following are equivalent (where \widehat{f} denotes fibrant replacement in $G\mathcal{J}\mathcal{S}$, which is unnecessary for GM):*

- (i). $(\widehat{f}g)_*^H: H_*((\widehat{f}X)^H; \mathbb{Q}) \rightarrow H_*((\widehat{f}Y)^H; \mathbb{Q})$ is an isomorphism for all H .
- (ii). $(\widehat{f}g)_*^H: \pi_*((\widehat{f}X)^H) \otimes \mathbb{Q} \rightarrow \pi_*((\widehat{f}Y)^H) \otimes \mathbb{Q}$ is an isomorphism for all H .
- (iii). $g \wedge \text{Id}: X \wedge S^0\mathbb{Q} \rightarrow Y \wedge S^0\mathbb{Q}$ is a π_* -isomorphism.
- (iv). $g: X \rightarrow Y$ is a $\pi_*^{\mathbb{Q}}$ -isomorphism.

Proof We have shown in Proposition 1.5.4 that the last two conditions are equivalent. Statements (ii) and (iv) are equivalent by [MM02, Chapter V, Proposition 3.2] (this is obvious for GM). The first two are well known to be equivalent, we are simply noting that our construction of $S^0\mathbb{Q}$ for G the trivial group gives $H\mathbb{Q}$, an Eilenberg-Mac Lane spectrum. ■

Lemma 2.2.2 *For any map $f: X \rightarrow Y$ of G -prespectra and any $H \subset G$, there are natural long exact sequences*

$$\begin{aligned} \cdots \rightarrow \pi_q^H(Ff) \otimes \mathbb{Q} \rightarrow \pi_q^H(X) \otimes \mathbb{Q} \rightarrow \pi_q^H(Y) \otimes \mathbb{Q} \rightarrow \pi_{q-1}^H(Ff) \otimes \mathbb{Q} \rightarrow \cdots, \\ \cdots \rightarrow \pi_q^H(X) \otimes \mathbb{Q} \rightarrow \pi_q^H(Y) \otimes \mathbb{Q} \rightarrow \pi_q^H(Cf) \otimes \mathbb{Q} \rightarrow \pi_{q-1}^H(X) \otimes \mathbb{Q} \rightarrow \cdots \end{aligned}$$

and the natural map $\nu: Ff \rightarrow \Omega Cf$ is a π_* -isomorphism.

Proof This follows from the fact that \mathbb{Q} is flat, (tensoring with \mathbb{Q} preserves exact sequences) and [MM02, Chapter III, Theorem 3.5] for orthogonal spectra ([MM02, Chapter IV, Remark 2.8] for GM). ■

Lemma 2.2.3 *The categories $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ and $GM_{\mathbb{Q}}$ are right proper.*

Proof We follow the proof of [MMSS01, 9.10] and show a stronger statement: in a pullback diagram as below, if β is a level fibration then r is a $\pi_*^{\mathbb{Q}}$ -isomorphism.

$$\begin{array}{ccc} W & \xrightarrow{\delta} & X \\ r \downarrow & \lrcorner & \downarrow \sim_{\mathbb{Q}} \\ Y & \xrightarrow{\beta} & Z \end{array}$$

Let β^{-1*} be the pullback of β over a point (the dual construction to Z/Y ; it is the pre-image of the basepoint of Z). This pullback is constructed levelwise and for each level V , $\delta(V)^{-1*} \cong \beta(V)^{-1*}$ (by simply writing down the definitions of these spaces), thus the map of spectra $\delta^{-1*} \rightarrow \beta^{-1*}$ is an isomorphism.

Now we use [Hat02, 4.65] to see that the fibre of a fibration of spaces is homotopy equivalent to the pre-image of the basepoint of the codomain. The map of spectra β is a level G -fibration, that is for each level V and subgroup H of G , $\beta(V)^H$ is a fibration. The fixed point functor $(-)^H$ is a right adjoint and $F(I, B^H) \cong F(I, B)^H$ for any spectrum B ([MM02, Chapter III, Lemma 1.6]). So it follows that the fibre of $\beta(V)$ is G -homotopy equivalent to $\beta(V)^{-1*} = (\beta^{-1*})(V)$ and thus we have a level G -equivalence between $F\beta$, the fibre of β , and β^{-1*} . Similarly we have a level G -equivalence between $F\delta$, the fibre of δ , and δ^{-1*} . Hence, we have a level G -equivalence (and thus a π_* -isomorphism) between $F\delta$ and $F\beta$.

We now apply the long exact sequence of rational homotopy groups of a fibration and the five-lemma to conclude that r is an E -equivalence. \blacksquare

Theorem 2.2.4 *For any X and Y , $[X, Y]_{\mathbb{Q}}^G$ is a rational vector space. If Z is an $S^0\mathbb{Q}$ -local object of $G\mathcal{J}\mathcal{S}$ then Z has rational homotopy groups. There is a natural isomorphism $[X, Y]_{\mathbb{Q}}^G \cong [X \wedge S^0\mathbb{Q}, Y \wedge S^0\mathbb{Q}]^G$.*

Proof The argument is the same for both model categories, so we use notation appropriate to $G\mathcal{J}\mathcal{S}$. For each integer n we have a self-map of $\widehat{f}\mathbb{S}$ which represents multiplication by n at the model category level, applying $(-) \wedge X$ we obtain a self-map of $\widehat{f}\mathbb{S} \wedge X$. Since this map is an isomorphism of rational homotopy groups it is a weak equivalence of $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ and so in the homotopy category of $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ we have an isomorphism $n: X \rightarrow X$ hence $[X, Y]_{\mathbb{Q}}^G$ is a rational vector space.

Let $\widehat{f}_{\mathbb{Q}}Z$ be the fibrant replacement of Z in $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$. The map $Z \rightarrow \widehat{f}_{\mathbb{Q}}Z$ is a rational equivalence between $S^0\mathbb{Q}$ -local objects and hence is a π_* -isomorphism. We can describe the homotopy groups of Z in terms of $[\Sigma^p G/H_+, Z]^G$ and $[F_q G/H_+, Z]^G$ for $p \geq 0$ and $q > 0$ by Lemma 1.3.6. The result then follows by the isomorphisms

$$[A, Z]^G \cong [A, \widehat{f}_{\mathbb{Q}}Z]^G \cong [A, Z]_{\mathbb{Q}}^G$$

which hold for any cofibrant G -spectrum A by Proposition 2.1.4.

Now we turn to the final part of this theorem. The map $Y \wedge S^0\mathbb{Q} \rightarrow \widehat{f}_{\mathbb{Q}}(Y \wedge S^0\mathbb{Q})$ is a $\pi_*^{\mathbb{Q}}$ -isomorphism between objects with rational homotopy groups, hence it is a π_* -isomorphism. For any G -spectrum X , $X \wedge S^0\mathbb{Q}$ is rationally equivalent to X . Combining these with Proposition 2.1.4 we obtain isomorphisms as below.

$$\begin{aligned} [X, Y]_{\mathbb{Q}}^G &\cong [X \wedge S^0\mathbb{Q}, Y \wedge S^0\mathbb{Q}]_{\mathbb{Q}}^G \\ &\cong [X \wedge S^0\mathbb{Q}, \widehat{f}_{\mathbb{Q}}(Y \wedge S^0\mathbb{Q})]^G \\ &\cong [X \wedge S^0\mathbb{Q}, Y \wedge S^0\mathbb{Q}]^G \end{aligned}$$

\blacksquare

Corollary 2.2.5 *If the spectrum X is H -compact in $G\mathcal{J}\mathcal{S}$ then it is H -compact in $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$. In particular, the generators of $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ are G -compact, or equivalently, \mathbb{S} is H -compact in $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ for all H .*

Proof Take a collection $\{Y_i\}_{i \in I}$ in $G\mathcal{J}\mathcal{S}$ and let $\widehat{f}_{\mathbb{Q}}$ be fibrant replacement in $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$. Then $[G/H_+, \bigvee_i Y_i]_{\mathbb{Q}}^G$ is isomorphic to

$$[G/H_+ \wedge S^0\mathbb{Q}, \bigvee_i Y_i \wedge S^0\mathbb{Q}]^G \cong [G/H_+ \wedge S^0\mathbb{Q}, \widehat{f}_{\mathbb{Q}}(\bigvee_i Y_i \wedge S^0\mathbb{Q})]^G.$$

Since $G/H_+ \rightarrow G/H_+ \wedge S^0\mathbb{Q}$ is a rational equivalence the above is isomorphic to

$$[G/H_+, \widehat{f}_{\mathbb{Q}}(\bigvee_i Y_i \wedge S^0\mathbb{Q})]^G \cong [G/H_+, (\bigvee_i Y_i) \wedge S^0\mathbb{Q}]^G.$$

Thus the result follows from the compactness of G/H_+ in $G\mathcal{J}\mathcal{S}$. ■

Proposition 2.2.6 *There is a Quillen equivalence*

$$\mathrm{Id} : G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+ \xrightleftharpoons{\quad} G\mathcal{J}\mathcal{S}_{\mathbb{Q}} : \mathrm{Id}.$$

Proof By Lemma 1.5.5 we see that $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ can be constructed by localising at $S^0\mathbb{Q}_+$, hence Theorem 2.1.12 gives us the result, see Remark 2.1.14. ■

Proposition 2.2.7 *The model structures $GM_{\mathbb{Q}}$ and $L_{NS^0\mathbb{Q}_+}GM$ are equal. The adjoint pair $(\mathbb{N}, \mathbb{N}^{\#})$ are a strong monoidal Quillen equivalence*

$$\mathbb{N} : G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+ \xrightleftharpoons{\quad} L_{NS^0\mathbb{Q}_+}GM : \mathbb{N}^{\#}$$

Proof We defined $GM_{\mathbb{Q}}$ as the localisation $L_{S_{\mathcal{M}}^0\mathbb{Q}}GM$. By Lemma 1.5.5 the cofibrant objects $NS^0\mathbb{Q}_+$ and $S_{\mathcal{M}}^0\mathbb{Q}$ are weakly equivalent. It follows therefore that a map is an $NS^0\mathbb{Q}_+$ -equivalence if and only if it is an $S_{\mathcal{M}}^0\mathbb{Q}$ -equivalence. Thus $L_{S_{\mathcal{M}}^0\mathbb{Q}}GM$ has the same weak equivalences and cofibrations as $L_{NS^0\mathbb{Q}_+}GM$. The second statement follows from Theorem 2.1.12, since $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+$ is the localisation of $G\mathcal{J}\mathcal{S}_+$ with respect to the cofibrant object $S^0\mathbb{Q}_+$. ■

2.3 $S_{\mathbb{Q}}$ -Modules

We give an alternative, but Quillen equivalent method of constructing a category of rational G -spectra. We do not consider orthogonal spectra in this section. The advantage to doing so is that every object in this new category will be fibrant. This is one of the technical requirements necessary to apply the results of [GS], see Remark 5.4.4.

Definition 2.3.1 We have a model category of modules over $S_{\mathbb{Q}}$ (see Definition 1.5.8) from [MM02, Chapter IV, Theorem 2.11] this will be written $S_{\mathbb{Q}}\text{-mod}$. This is a proper closed symmetric monoidal model category.

Lemma 2.3.2 *A map is an $S_{\mathcal{M}}^0\mathbb{Q}$ -equivalence if and only if it is a $\widehat{c}S_{\mathbb{Q}}$ -equivalence.*

Proof The result follows from the zig-zag of weak equivalences (which exists for any spectrum X): $X \wedge S_{\mathcal{M}}^0\mathbb{Q} \leftarrow \widehat{c}X \wedge S_{\mathcal{M}}^0\mathbb{Q} \rightarrow \widehat{c}X \wedge S_{\mathbb{Q}} \leftarrow \widehat{c}X \wedge \widehat{c}S_{\mathbb{Q}} \rightarrow X \wedge \widehat{c}S_{\mathbb{Q}}$. ■

Lemma 2.3.3 *All $S_{\mathbb{Q}}$ -modules are $S_{\mathcal{M}}^0\mathbb{Q}$ -local and thus all $S_{\mathbb{Q}}$ -modules have rational homotopy groups.*

Proof Since fibrations of $S_{\mathbb{Q}}$ -modules are defined in terms of their underlying S -modules and all S -modules are fibrant, we must show that $[A, M]^G = 0$ for any $S_{\mathcal{M}}^0\mathbb{Q}$ -acyclic A and any $S_{\mathbb{Q}}$ -module M . We adapt the following argument from [Ada74, 13.1]. Take $[f] \in [A, M]$ then $[f]$ is zero if and only if $f^*: [M, M] \rightarrow [A, M]$ is the zero map.

$$\begin{array}{ccc} [M, M]^G & \xrightarrow{f^*} & [A, M]^G \\ \nu^* \downarrow & & \uparrow (\text{Id} \wedge \eta)^* \\ [M \wedge^L S_{\mathbb{Q}}, M]^G & \xrightarrow{(f \wedge \text{Id})^*} & [A \wedge^L S_{\mathbb{Q}}, M]^G \end{array}$$

Since $A \wedge S_{\mathcal{M}}^0\mathbb{Q}$ is acyclic, it follows by Lemma 2.3.2 that $A \wedge^L S_{\mathbb{Q}} = \widehat{c}A \wedge \widehat{c}S_{\mathbb{Q}}$ is acyclic, hence $[A \wedge^L S_{\mathbb{Q}}, M]^G = 0$. From this it is clear that $f^* = 0$ and M is an $S^0\mathbb{Q}$ -local object. ■

Theorem 2.3.4 *There is a strong symmetric monoidal Quillen equivalence:*

$$S_{\mathbb{Q}} \wedge (-) : G\mathcal{M}_{\mathbb{Q}} \rightleftarrows S_{\mathbb{Q}}\text{-mod} : U.$$

Proof The above functors form a strong monoidal Quillen pair (with the usual structure on $G\mathcal{M}$), this is standard and comes from the construction of a monoidal model structure on $S_{\mathbb{Q}}\text{-mod}$. Now consider the localised case, since the cofibrations are unaffected by localisation, $S_{\mathbb{Q}} \wedge (-) : G\mathcal{M}_{\mathbb{Q}} \rightarrow S_{\mathbb{Q}}\text{-mod}$ preserves cofibrations. Consider an acyclic rational cofibration $X \rightarrow Y$, we know that $S_{\mathbb{Q}} \wedge (-)$ applied to this gives a cofibration, we must check that it is also a π_* -isomorphism. We see that $X \wedge S_{\mathcal{M}}^0\mathbb{Q} \rightarrow Y \wedge S_{\mathcal{M}}^0\mathbb{Q}$ is a cofibration and a π_* -isomorphism, so in turn $X \wedge S_{\mathcal{M}}^0\mathbb{Q} \wedge S_{\mathbb{Q}} \rightarrow Y \wedge S_{\mathcal{M}}^0\mathbb{Q} \wedge S_{\mathbb{Q}}$ is an h -cofibration and a π_* -isomorphism. This proves that $X \wedge S_{\mathbb{Q}} \rightarrow Y \wedge S_{\mathbb{Q}}$ is a $\pi_*^{\mathbb{Q}}$ -isomorphism between $S_{\mathbb{Q}}$ -modules, which we know have rational homotopy groups and thus this map is a π_* -isomorphism.

We prove that this is a Quillen equivalence using the characterisation of Quillen equivalences from Theorem 3.2.4. The right adjoint preserves and detects all weak equivalences, since a $\pi_*^{\mathbb{Q}}$ -isomorphism between objects with rational homotopy groups is a π_* -isomorphism. So to finish this proof we must show that the map $X \rightarrow S_{\mathbb{Q}} \wedge X$ is a rational equivalence for all cofibrant S -modules X . This follows since smashing with a cofibrant object will preserve the $\pi_*^{\mathbb{Q}}$ -isomorphism $S \rightarrow S_{\mathbb{Q}}$. ■

Remark 2.3.5 This model category is proper by [MM02, Chapter IV 2.11].

Chapter 3

Splitting Rational G -Spectra

Idempotents of the Burnside ring split the homotopy category of G -spectra, we provide a model category level version of this in the second section (Theorem 3.2.4). To prove this theorem we must use some classical results on G -spectra which we give in Section 3.1. We show that this splitting applies to all our categories of rational G -spectra in Section 3.3. By considering families of subgroups in the final section of this chapter we obtain a particularly nice form of the splitting (Theorem 3.4.14). That is, we use [MM02, Chapter IV, Section 6] to understand the split pieces of the category of G -spectra in this case. The arguments in this chapter will apply equally well to $G\mathcal{I}\mathcal{S}_{\mathbb{Q}}$, $G\mathcal{I}\mathcal{S}_{\mathbb{Q}}^+$ and $GM_{\mathbb{Q}}$. For definiteness we work with $G\mathcal{I}\mathcal{S}$ in Section 3.1, Section 3.2 and Section 3.4. We will only consider other categories of equivariant spectra in Section 3.3.

3.1 Equivariant Stable Homotopy Theory

We prove a medley of basic results about spectra and equivariant spectra. These are the tools we will use to prove the splitting theorem in the following section. It is not easy to find the proofs of these results, which is why they are included here. The most important of these results are Proposition 3.1.5 and Proposition 3.1.10.

Lemma 3.1.1 *In $G\mathcal{I}\mathcal{S}$, let $f: E \rightarrow F$ be a map between cofibrant objects, then a map $g: X \rightarrow Y$ which is an E -equivalence and a Cf -equivalence is also an F -equivalence.*

Proof Consider the map of cofibre sequences

$$\begin{array}{ccccc} E \wedge X & \longrightarrow & F \wedge X & \longrightarrow & Cf \wedge X \\ \downarrow \text{Id}_E \wedge g & & \downarrow \text{Id}_F \wedge g & & \downarrow \text{Id}_{Cf} \wedge g \\ E \wedge Y & \longrightarrow & F \wedge Y & \longrightarrow & Cf \wedge Y \end{array}$$

this gives a map of long exact sequences of homotopy groups [MM02, Chapter III, Theorem 3.5] and we apply the five lemma. ■

Remark 3.1.2 In fact, the above proof shows that for a map $f: E \rightarrow F$ of cofibrant objects, one only has to check that two out of three of the maps $\text{Id}_E \wedge g$, $\text{Id}_F \wedge g$ and $\text{Id}_{Cf} \wedge g$ are weak equivalences to conclude that all three are weak equivalences. Also note that if E and F are cofibrant, then so is Cf . A wedge $E \vee F$ of cofibrant objects comes with a cofibre sequence $E \rightarrow E \vee F \rightarrow (E \wedge I) \vee F$. Since $(E \wedge I) \vee F$ is weakly equivalent to F , a map is an $E \vee F$ -equivalence if and only if it is an E -equivalence and an F -equivalence.

Since $A(G) := [\mathbb{S}, \mathbb{S}]^G$ we know that $A(G) \otimes \mathbb{Q} = [\mathbb{S}, \mathbb{S}]_{\mathbb{Q}}^G$. So $[a]$, an element of the Burnside ring, can be represented by a self map a of $\widehat{f}_{\mathbb{Q}}\mathbb{S}$ and hence for any spectrum X , we have $\text{Id}_X \wedge a$, a self-map of $X \wedge \widehat{f}_{\mathbb{Q}}\mathbb{S}$.

Definition 3.1.3 Let e be a self-map of $\widehat{f}_{\mathbb{Q}}\mathbb{S}$ such that $[e] \in A(G) \otimes \mathbb{Q}$ is an idempotent (thus $e \circ e \simeq e$). We define eX to be the homotopy colimit of

$$X \wedge \widehat{f}_{\mathbb{Q}}\mathbb{S} \xrightarrow{\text{Id} \wedge e} X \wedge \widehat{f}_{\mathbb{Q}}\mathbb{S} \xrightarrow{\text{Id} \wedge e} X \wedge \widehat{f}_{\mathbb{Q}}\mathbb{S} \xrightarrow{\text{Id} \wedge e} X \wedge \widehat{f}_{\mathbb{Q}}\mathbb{S} \xrightarrow{\text{Id} \wedge e} \dots;$$

fibrant replacement $\mathbb{S} \rightarrow \widehat{f}_{\mathbb{Q}}\mathbb{S}$ provides a map $X \rightarrow eX$.

Remark 3.1.4 An idempotent $e \in A(G) \otimes \mathbb{Q}$ has a support $S \subseteq S_f G$. Given a subgroup H of G (with inclusion ι_H), there is an idempotent $\iota_H^*(e) \in A(H) \otimes \mathbb{Q}$. This idempotent is supported on the set $\iota_H^*(S) := \{K \leq H \mid K \in S\}$. So considering $\iota_H^*(e)$ as a continuous map $\mathcal{F}H/H \rightarrow \mathbb{Q}$, $\iota_H^*(e)(K)_H \neq 0$ exactly when $K \in S$. That is, $\iota_H^*(e)$ is non-zero on the H -conjugacy class of a subgroup K of H if and only if e is non-zero on the G -conjugacy class of K .

The following well-known result can be described as proving that the homogenous spaces G/H_+ are small in the homotopy category of spectra with respect to the h -cofibrations. We give some comments on this after the statement and proof of the result. This type of result is proven in greater generality in [Hov08, Section 4].

Proposition 3.1.5 *For a sequential colimit diagram $f_i: X_i \rightarrow X_{i+1}$ there is an isomorphism of groups $\pi_*^H(\text{HoColim}_i X_i) \cong \text{Colim}_i \pi_*^H(X_i)$.*

Proof The homotopy colimit is formed by replacing the diagram $f_i: X_i \rightarrow X_{i+1}$ by a sequence $g_i: Y_i \rightarrow Y_{i+1}$ of h -cofibrations, we can now consider this as a sequence of h -cofibrations of prespectra. In order to take H -fixed points we must first apply ι_H^* . Recall that ι_H^* is both a right and left adjoint and hence commutes with all limits and colimits, it preserves h -cofibrations and we have the relation $\iota_H^*F(X, Y) \cong F(\iota_H^*X, \iota_H^*Y)$. Thus we can suppress the notation for ι_H^* and this will cause no difficulty. Fix some subgroup H , then we have the homotopy group $\text{Colim}_V \pi_q^H(\Omega^V(\text{Colim}_i Y_i)(V))$. Since colimits of prespectra are created levelwise, this is equal to $\text{Colim}_V \pi_q(\Omega^V \text{Colim}_i (Y_i(V))^H)$. Now we are working in H -spaces and S^V (or for the negative case $S^{V-\mathbb{R}^q}$) is a finite H -CW complex. So there are subgroups K_α of H , where α runs over some finite set such that $S^V \cong \text{Colim}_\alpha G/K_\alpha \wedge S^{n_\alpha}$. Thus the standard adjunctions give

$$\begin{aligned} F(S^V, \text{Colim}_i Y_i(V))^H &\cong \text{Lim}_\alpha F(S^{n_\alpha}, \text{Colim}_i Y_i(V))^{K_\alpha} \\ &\cong \text{Lim}_\alpha F(S^{n_\alpha}, \text{Colim}_i Y_i(V)^{K_\alpha}). \end{aligned}$$

Hence we can write $\pi_q(\Omega^V \operatorname{Colim}_i(Y_i(V))^H) \cong \operatorname{Lim}_\alpha \pi_{q+n_\alpha}(\operatorname{Colim}_i(Y_i(V))^{K_\alpha})$. Now we use the fact that h -cofibrations are levelwise monomorphisms so that our colimit commutes with taking fixed points. Furthermore $(-)^{K_\alpha}$ takes h -cofibrations of H -spaces to cofibrations of spaces, thus since S^{q+n_α} is a compact space

$$\pi_{q+n_\alpha}(\operatorname{Colim}_i(Y_i(V)^{K_\alpha})) \cong \operatorname{Colim}_i \pi_{q+n_\alpha}(Y_i(V)^{K_\alpha}).$$

Since sequential colimits commute with finite limits and colimits we can repack all of the above to obtain our stated result. ■

This result also holds for GM with almost the same proof, with one extra point of justification. A colimit of h -cofibrations of inclusion prespectra is an inclusion prespectrum. Spectrification for inclusion prespectra is given by $LX(V) = \operatorname{Colim}_{W \supset V} \Omega^{W-V} X(W)$. Hence, by the arguments in the above proof, colimits of h -cofibrations of spectra are given by the levelwise colimit.

Remark 3.1.6 Our proof above actually holds for filtered colimits, rather than just sequential colimits. In the proof of Lemma 1.3.10 we described a coproduct in terms of a filtered colimit. One can make the converse construction and describe a filtered colimit as a cofibre of coproducts. Take a filtered colimit diagram $\{Y_i\}_{i \in I}$ such that the maps $Y_i \rightarrow Y_j$ are all h -cofibrations. For $i \in I$ let $I(i)$ be the subset of those $j \in I$ such that there is a map $Y_i \rightarrow Y_j$. We define a map $f: \bigvee_{i \in I} Y_i \rightarrow \bigvee_{i \in I} Y_i$ by sending $Y_i \rightarrow \bigvee_{j \in I(i)} Y_j$ and including this in $\bigvee_{i \in I} Y_i$. Since the homotopy category is additive we can take a representative $g: X \rightarrow Z$ for the homotopy class of maps $[1 - f]: \widehat{cf} \bigvee_{i \in I} Y_i \rightarrow \widehat{cf} \bigvee_{i \in I} Y_i$. The cofibre of g is weakly equivalent to $\operatorname{Colim}_i Y_i$ by looking at the long exact sequence of a cofibration and noting that $(1 - f)_*$ is injective on homotopy groups. So in the homotopy category of G -spectra, the notion of small and compact are the same, though one must be careful about constructing filtered colimits. Replacing colimits by a cofibre of coproducts is a standard construction, see [HPS97, Definition 2.23].

Corollary 3.1.7 *Let e be an idempotent of the rational Burnside ring and let X be any orthogonal spectrum. Then for any subgroup H of G we have the inclusion $\iota_H: H \rightarrow G$ and isomorphisms $\pi_*^H(eS \wedge X) \otimes \mathbb{Q} \cong \pi_*^H(eX) \otimes \mathbb{Q} \cong \iota_H^*(e)\pi_*^H(X) \otimes \mathbb{Q}$.*

Proof The first isomorphism follows from Proposition 1.4.10 the rest will follow from the previous result. Note that the sequential colimit of an idempotent is isomorphic to the image of the idempotent. Since we have suppressed ι_H^* in our notation for homotopy groups, we must account for its action on the idempotent hence the term $\iota_H^*(e)$. ■

Proposition 3.1.8 *The map $X \rightarrow X \amalg X \rightarrow eX \amalg (1 - e)X$ is a rational equivalence for any X and any idempotent e of the Burnside ring.*

Proof They certainly have isomorphic rational homotopy groups and examination of the maps involved shows that the result is true. ■

Proposition 3.1.9 *Let X be a cofibrant orthogonal G -spectrum. Let $[e]$ be an idempotent of the rational Burnside ring of G with support $S \subseteq S_f G$. Then for any*

$L \in S$, $\Phi^L(eX)$ is rationally equivalent to $\Phi^L(X)$ as non-equivariant spectra, otherwise $\Phi^L(eX)$ is non-equivariantly rationally acyclic.

Proof We perform much of the proof in the category of rational orthogonal L -spectra, that is, we apply the forgetful functor ι_L^* to all the spectra involved, (ι_L is the inclusion of L in G). This functor ι_L^* preserves fibrations, cofibrations, weak equivalences and more ([MM02, Chapter V, Lemma 2.2]) so this presents us with no difficulty. In general when we write $\Phi^H(X)$ for a G -spectrum X and H a subgroup of G , we mean $\Phi^H(\iota_H^* X)$. Also note that $\iota_L^*(eX)$ is weakly equivalent to $\iota_L^*(e)\iota_L^*(X)$ as rational L -spectra.

We now apply Proposition 1.4.10 to obtain a weak equivalence (of L -spectra) $eX \rightarrow e\mathbb{S} \wedge X$ between cofibrant objects and hence (by Ken Brown's lemma) we have a weak equivalence of non-equivariant spectra $\Phi^L(eX) \rightarrow \Phi^L(e\mathbb{S} \wedge X)$. Using [MM02, Chapter V, Section 4] we have a further weak equivalence

$$\Phi^L(e\mathbb{S}) \wedge \Phi^L(X) \rightarrow \Phi^L(e\mathbb{S} \wedge X)$$

and we know that Φ^L commutes with colimits of h -cofibrations so that $\Phi^L(e\mathbb{S}) \simeq \Phi^L(e)\Phi^L\mathbb{S}$. Since \mathbb{S} is the suspension spectrum of S^0 which is G -fixed, $\Phi^L\mathbb{S} \cong \mathbb{S}$, now we only have to understand the map $\Phi^L(e)$. The construction of tom Dieck's isomorphism

$$A(G) \otimes \mathbb{Q} \cong C(\mathcal{F}G/G, \mathbb{Q})$$

takes a map of spectra f to the map $H \mapsto \deg \Phi^H(f)$, where degree has the usual algebraic topology definition in terms of homology, which coincides with homotopy (since we are working rationally and stably). Thus

$$\pi_*(\Phi^L(e)\mathbb{S}) \cong \Phi^L(e)_*\pi_*(\mathbb{S}) = \deg \Phi^L(e)\pi_*(\mathbb{S})$$

which is either zero or $\pi_*(\mathbb{S})$ according to whether or not $L \in S$. When $L \notin S$ we are smashing $\Phi^L(X)$ with an acyclic object, hence (since Φ^L preserves cofibrations and X is cofibrant) $\Phi^L(eX)$ is an acyclic non-equivariant orthogonal spectrum. For the other case we have a weak equivalence of non-equivariant spectra $\mathbb{S} \rightarrow \Phi^L(e)\mathbb{S}$ (since $\Phi^L(e)$ is a π_* -isomorphism) and so we have our result. ■

The following theorem is given in [May96, Chapter XVI, Theorem 6.4] we provide a 'folk-proof' of this 'folk-theorem' in the language of orthogonal G -spectra.

Proposition 3.1.10 *Let $f: X \rightarrow Y$ be a map of cofibrant G -equivariant orthogonal spectra, then $\Phi^H f$ is a weak equivalence of non-equivariant orthogonal spectra for all $H \leq G$ if and only if f is a weak equivalence of G -equivariant orthogonal spectra.*

Proof We must use $\iota_H^*: G\mathcal{J}\mathcal{S} \rightarrow H\mathcal{J}\mathcal{S}$ in order to apply Φ^H to a G -equivariant spectrum. As with the previous proposition, we omit ι_H^* -notation in the following proof. The 'if' part is immediate by Ken Brown's lemma applied to Φ^H (since it preserves (acyclic) cofibrations). The converse begins by noting that Φ^H preserves cofibre sequences, so it suffices to prove (for cofibrant Z) that if $\Phi^H(Z)$ is acyclic for all H then Z is acyclic as a G -spectrum. We use [MM02, Chapter V, Proposition 4.17] to replace $\Phi^H(Z)$ by the weakly equivalent object $(\widehat{f}(Z \wedge E\mathcal{F}_H))^H$. The family

\mathcal{F}_H is the collection of subgroups of H which do not contain H : it is the family of proper subgroups of H .

We begin an inductive argument at $\{e\}$, where $\widetilde{E\mathcal{F}_{\{e\}}} = S^0$. Since $\Phi^{\{e\}}Z$ is acyclic, $\pi_*((\widehat{f}Z)^{\{e\}}) = \pi_*^{\{e\}}(\widehat{f}Z) = 0$. Hence Z is contractible as an $\{e\}$ -spectrum. Now take a subgroup H and assume inductively that Z is contractible as a K -spectrum for all strict subgroups K of H . We will show that $Z \wedge \widetilde{E\mathcal{F}_H}$ and $Z \wedge E\mathcal{F}_{H+}$ are contractible H -spectra and apply Lemma 3.1.1 to see that Z will be H contractible.

Consider $(\widehat{f}(Z \wedge \widetilde{E\mathcal{F}_H}))^K$, for a strict subgroup K of H . By inductive assumption Z is K -acyclic, hence so is $Z \wedge \widetilde{E\mathcal{F}_H}$, thus $\pi_*^K(Z \wedge \widetilde{E\mathcal{F}_H}) = 0$. We started by assuming that $\Phi^H(Z)$ is acyclic, hence $\pi_*^H(Z \wedge \widetilde{E\mathcal{F}_H}) = 0$. We have completed the first half of our inductive step. We will now prove that $[Z \wedge E\mathcal{F}_{H+}, M]^H = 0$ for any H -spectrum M and thus $Z \wedge E\mathcal{F}_{H+}$ will be acyclic as an H -spectrum. The space $E\mathcal{F}_{H+}$ is made from cells of type H/K for K a strict subgroup of H , $E\mathcal{F}_{H+} = \text{Colim}_\alpha H/K_+^\alpha \wedge S^{n_\alpha}$. It follows by standard manipulations that

$$[Z \wedge E\mathcal{F}_{H+}, M]^H \cong \text{Lim}_\alpha [Z \wedge S^{n_\alpha}, M]^{K^\alpha}$$

using the fact that $H/K_+^\alpha \wedge Z \cong H_+ \wedge_{K^\alpha} Z$ as H -spectra. This last term is zero since Z is acyclic as a K^α -spectrum. ■

Note that in the above we use induction on the poset of closed subgroups of a compact Lie group. Such inductive arguments are valid since there are no infinite descending chains of subgroups in a compact Lie group.

Corollary 3.1.11 *The above proposition also holds in the rationalised case: $f: X \rightarrow Y$ is a rational equivalence of G -spectra if and only if $\Phi^H f$ is a rational equivalence of non-equivariant spectra for all subgroups H of G .*

Proof We temporarily let the rational sphere spectrum of G -spectra as $S^0\mathbb{Q}_G$, then we see that $\Phi^H(S^0\mathbb{Q}_G) = S^0\mathbb{Q}_{\{e\}}$ for all H . The result then follows by the usual properties of Φ^H (see [MM02, V section 4]). ■

3.2 The Splitting

We now prove the most important result of this chapter, indeed, the most important result of Part I: Theorem 3.2.4.

Definition 3.2.1 Recall the definition of the **product model category** from [Hov99, Example 1.1.6]. Given model categories M_1 and M_2 we can put a model category structure on $M_1 \times M_2$. A map (f_1, f_2) is a cofibration, weak equivalence or fibration if and only if f_1 is so in M_1 and f_2 is so in M_2 . Similarly a finite product of model categories has a model structure where a map is a cofibration, weak equivalence or fibration if and only if each of its factors is so.

Remark 3.2.2 If M_1 and M_2 both satisfy any of the following: left properness, right properness, the pushout product axiom, the monoid axiom or cofibrant generation, then so does $M_1 \times M_2$.

Proposition 3.2.3 *If E and F are cofibrant orthogonal G -spectra or cofibrant G -spaces, then there is a strong monoidal Quillen adjunction*

$$\Delta : G\mathcal{J}\mathcal{S} \rightleftarrows L_E G\mathcal{J}\mathcal{S} \times L_F G\mathcal{J}\mathcal{S} : \prod$$

(we follow the usual convention of placing the left adjoint on top).

Proof Take a map $f : X \rightarrow Y$ in $G\mathcal{J}\mathcal{S}$, then $\Delta(f) = (f, f) : (X, X) \rightarrow (Y, Y)$. For a map $(a, b) : (A, B) \rightarrow (C, D)$, $\prod(a, b)$ is given by $a \prod b : A \prod B \rightarrow C \prod D$. That these are an adjoint pair is easy to see since all of the isomorphisms and equalities below are natural.

$$\begin{aligned} L_E G\mathcal{J}\mathcal{S} \times L_F G\mathcal{J}\mathcal{S}((X, X), (A, B)) &= L_E G\mathcal{J}\mathcal{S}(X, A) \times L_F G\mathcal{J}\mathcal{S}(X, B) \\ &= G\mathcal{J}\mathcal{S}(X, A) \times G\mathcal{J}\mathcal{S}(X, B) \\ &\cong G\mathcal{J}\mathcal{S}(X, A \prod B) \end{aligned}$$

If f is a (acyclic) cofibration of $G\mathcal{J}\mathcal{S}$, then it is clear that (f, f) is a (acyclic) cofibration of $L_E G\mathcal{J}\mathcal{S} \times L_F G\mathcal{J}\mathcal{S}$, thus Δ is a left Quillen functor. It is easy to see that this is a strong monoidal adjunction. ■

It is clear that this result can be extended to any finite product of localisations. Now we turn to a case when this adjunction is a Quillen equivalence. We state the following theorem rationally, but one can easily see that a similar result will hold in any localised case, as well as for $G\mathcal{J}\mathcal{S}$ itself.

Theorem 3.2.4 *Let $\{E_i\}_{i \in I}$ be a finite collection of cofibrant orthogonal G -spectra or G -spaces. If $E_i \wedge E_j$ is rationally acyclic for $i \neq j$ and $\bigvee_{i \in I} E_i$ is rationally equivalent to \mathbb{S} then we have a strong monoidal Quillen equivalence*

$$\Delta : G\mathcal{J}\mathcal{S}_{\mathbb{Q}} \rightleftarrows \prod_{i \in I} L_{E_i} G\mathcal{J}\mathcal{S}_{\mathbb{Q}} : \prod.$$

Proof Proposition 3.2.3 implies that this is a strong monoidal Quillen pair, so we must show that this is a Quillen equivalence. We use the characterisation of Quillen equivalences of [HSS00, Lemma 4.1.7], we must show that:

- (i). \prod detects and preserves weak equivalences between fibrant objects,
- (ii). $X \rightarrow \prod \Delta X \rightarrow \prod \widehat{f} \Delta X$ is a $\pi_*^{\mathbb{Q}}$ -isomorphism for all cofibrant X in $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$

where \widehat{f} denotes fibrant replacement in $\prod_{i \in I} L_{E_i} G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$. The first statement follows easily, take a map $f : A \rightarrow B$ between fibrant objects in $\prod_{i \in I} L_{E_i} G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$. The map f is a weak equivalence exactly when each factor f_i is a rational E_i -equivalence. Recall that a rational E_i -equivalence between $E_i \wedge S^0 \mathbb{Q}$ -local objects is a π_* -isomorphism, hence each f_i is a π_* -isomorphism. Thus f is a weak equivalence if and only if $\prod_i f_i$ is a π_* -isomorphism, (since homotopy groups commute with products).

Now we must show that $X \rightarrow \prod \widehat{f} \Delta X$ is a weak equivalence for all cofibrant X . This is equivalent to proving that $\prod \widehat{f} \Delta X \rightarrow \widehat{f}_{E_i} X$ is a rational equivalence for each i , where \widehat{f}_{E_i} denotes fibrant replacement in $L_{E_i} G\mathcal{J}\mathcal{S}\mathcal{Q}$. Since this is a finite product, it is weakly equivalent to $\bigvee_{j \in I} \widehat{f}_{E_j} X$. So we will prove that $\bigvee_{j \in I} \widehat{f}_{E_j} X \rightarrow \widehat{f}_{E_i} X$ is a rational equivalence. Since $\bigvee_{i \in I} E_i$ is rationally equivalent to \mathbb{S} we can use Remark 3.1.2 repeatedly to see that a map f in $G\mathcal{J}\mathcal{S}\mathcal{Q}$ is a rational equivalence if and only if $f \wedge \text{Id}_{E_i}$ is a rational equivalence for all $i \in I$. This amounts to showing that $\widehat{f}_{E_j} X \wedge E_i$ is rationally acyclic when $i \neq j$.

Since $\bigvee_{i \in I} E_i$ and \mathbb{S} are rationally equivalent, they are isomorphic in $\text{Ho } G\mathcal{J}\mathcal{S}\mathcal{Q}$. Thus the idempotent map $p_i: \bigvee_{i \in I} E_i \rightarrow E_i \rightarrow \bigvee_{i \in I} E_i$ gives an idempotent map of \mathbb{S} in $\text{Ho } G\mathcal{J}\mathcal{S}\mathcal{Q}$. Choosing a representative for this map $e_i: \widehat{f}_{\mathbb{Q}} \mathbb{S} \rightarrow \widehat{f}_{\mathbb{Q}} \mathbb{S}$ we obtain $e_i \widehat{f}_{\mathbb{Q}} \mathbb{S}$. Since p_i is a cofibration, $p_i \bigvee_{i \in I} E_i$ is rationally equivalent to E_i . Hence so is $(\widehat{f}_{\mathbb{Q}} p_i)(\widehat{f}_{\mathbb{Q}} \bigvee_{i \in I} E_i)$. We can construct a homotopy commuting square

$$\begin{array}{ccc} \widehat{f}_{\mathbb{Q}} \mathbb{S} & \xrightarrow{e_i} & \widehat{f}_{\mathbb{Q}} \mathbb{S} \\ \sim \downarrow & & \sim \downarrow \\ \widehat{f}_{\mathbb{Q}} \bigvee_{i \in I} E_i & \xrightarrow{\widehat{f}_{\mathbb{Q}} p_i} & \widehat{f}_{\mathbb{Q}} \bigvee_{i \in I} E_i \end{array}$$

such that the vertical maps are weak equivalences (since $\widehat{f}_{\mathbb{Q}} \mathbb{S}$ and $\bigvee_{i \in I} E_i$ are rationally equivalent). Now we apply Lemma 1.4.9 to conclude that $e_i \widehat{f}_{\mathbb{Q}} \mathbb{S}$ is rationally equivalent to E_i . We now see that a rational E_i -equivalence is the same as a rational $e_i \widehat{f}_{\mathbb{Q}} \mathbb{S}$ -equivalence. Thus, by Corollary 3.1.7, $f: X \rightarrow Y$ is a rational E_i -equivalence if and only if $e_i^* \pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all H . So we have isomorphisms $e_i^* \pi_*^H(\widehat{f}_{E_i} X) \otimes \mathbb{Q} \rightarrow e_i^* \pi_*^H(X) \otimes \mathbb{Q}$ and $\pi_*^H(E_i \wedge X) \otimes \mathbb{Q} \cong e_i^* \pi_*^H(X) \otimes \mathbb{Q}$.

The projection $\bigvee_{j \in I} E_j \rightarrow E_i$ is a rational E_i equivalence (since $E_i \wedge E_j$ is rationally acyclic for $i \neq j$). Now we use the fact that $\widehat{f}_{E_i} X$ is E_i -local to obtain isomorphisms

$$\pi_n^H(\widehat{f}_{E_i} X) \otimes \mathbb{Q} \cong [\Sigma^n \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]^H \otimes \mathbb{Q} \cong [\Sigma^n E_i, \iota_H^* \widehat{f}_{E_i} X]^H \otimes \mathbb{Q} \cong [e_i \Sigma^n \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]^H \otimes \mathbb{Q}$$

Now we apply [MM02, Chapter III, Theorems 2.4 and 2.7], since $e_i \Sigma^n \mathbb{S}$ is a colimit of cofibrant objects it is non-degenerately based, thus we obtain the Lim^1 exact sequence of pointed sets below.

$$* \rightarrow \text{Lim}^1 [\Sigma^{n+1} \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]_l^H \rightarrow [e_{\mathcal{F}} \Sigma^n \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]_l^H \rightarrow \text{Lim} [\Sigma^n \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]_l^H \rightarrow *$$

Since e_i^* (that is, e_i acting on the first factor) is an idempotent on sets of maps in the rational homotopy category and $[\Sigma^{n+1} \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]_l^H \cong [\Sigma^{n+1} \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]_{\mathbb{Q}}^H$ the tower it creates satisfies the Mittag-Leffler condition ([Wei94, Definition 3.5.6]) hence the Lim^1 term is zero. So we see that

$$[e_i \Sigma^n \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]_l^H \cong e_i^* [\Sigma^n \mathbb{S}, \iota_H^* \widehat{f}_{E_i} X]_l^H$$

(a limit of idempotent maps is equivalent to taking the image). Now consider the action of idempotents on maps in the homotopy category: $e_i^*[f] = [f] \wedge [e_i]$ and $(e_j)_*[f] =$

$[f] \wedge [e_j]$, hence $(e_j)_*(e_i)^* = (e_i)_*e_j^* = 0$. We can now finish our argument as follows:

$$\pi_*^H(\widehat{f}_{E_i}X \wedge E_j) \otimes \mathbb{Q} \cong (e_j)_*(e_i)^*\pi_*^H(\widehat{f}_{E_i}X) \otimes \mathbb{Q} = 0.$$

■

Corollary 3.2.5 *Finite orthogonal idempotent decompositions of the unit of $A(G)$ correspond to finite splittings of $G\mathcal{J}\mathcal{S}$. The same statement also holds for $A(G) \otimes \mathbb{Q}$ and $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$.*

Proof Let $1 \in A(G)$ be the sum of (a finite collection of) idempotents e_i , such that $e_i e_j = 0$ whenever $i \neq j$, then $G\mathcal{J}\mathcal{S}$ splits as the product of the localised categories $L_{e_i\mathbb{S}}G\mathcal{J}\mathcal{S}$. Conversely, if $G\mathcal{J}\mathcal{S}$ splits as the product of localised categories $L_{E_i}G\mathcal{J}\mathcal{S}$ then $A(G) \cong \bigoplus_i [\mathbb{S}, \mathbb{S}]_{E_i}^G$. Hence, for each i , we have an idempotent element e_i which is the unit map in factor i and the trivial map elsewhere. ■

Remark 3.2.6 Let E_i be a collection of spectra satisfying the assumptions of the splitting theorem with corresponding idempotents e_i . Let X and Y be spectra, then $[X, Y]^G$ is an $A(G)$ -module. If Y is E_i -local then $[X, Y]^G$ is isomorphic to $e_i[X, Y]^G$ (and is also isomorphic to maps in the homotopy category of $L_{E_i}G\mathcal{J}\mathcal{S}$). If M is an $A(G)$ -module then M with e_i inverted is given by $e_i M$ (since e_i is an idempotent). Thus we can say that localisation at E_i inverts $e_i \in A(G)$. Equally, rationalisation inverts the primes in $A(G)$. This explains why our two kinds of localisation (rationalisation and splitting) behave the same: in each case we are simply inverting elements of $A(G)$.

3.3 Comparisons

We show that the splitting theorem for $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ implies the corresponding splitting for $G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+$, $GM_{\mathbb{Q}}$ and $S_{\mathbb{Q}}\text{-mod}$. We let \widehat{c}_+ denote cofibrant replacement in $G\mathcal{J}\mathcal{S}_+$.

Theorem 3.3.1 *Let $\{E_i\}_{i \in I}$ be a finite collection of cofibrant orthogonal G -spectra or G -spaces. If $E_i \wedge E_j$ is rationally acyclic for $i \neq j$ and $\bigvee_{i \in I} E_i$ is rationally equivalent to \mathbb{S} then we have a strong monoidal Quillen equivalence*

$$\Delta : G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+ \rightleftarrows \prod_{i \in I} L_{\widehat{c}_+ E_i} G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+ : \prod.$$

Proof We can replace the collection $\{E_i\}$ by their positive cofibrant replacements $\{\widehat{c}_+ E_i\}$. It is clear that $\bigvee_{i \in I} \widehat{c}_+ E_i$ is rationally equivalent to \mathbb{S} and $\widehat{c}_+ E_i \wedge \widehat{c}_+ E_j$ is rationally acyclic for $i \neq j$. Now we can compare the statement above to that for $G\mathcal{J}\mathcal{S}$ by using Remark 2.1.14. Thus we see that the category $L_{\widehat{c}_+ E_i} G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^+$ is Quillen equivalent to $L_{E_i} G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$, hence the splitting applies to the positive stable case. ■

Theorem 3.3.2 *Let $\{E_i\}_{i \in I}$ be a finite collection of positive cofibrant orthogonal G -spectra or G -spaces. Assume that $E_i \wedge E_j$ is rationally acyclic for $i \neq j$ and $\bigvee_{i \in I} E_i$*

is rationally equivalent to \mathbb{S} . We have the following pair of commutative (see proof) diagrams of strong symmetric monoidal Quillen equivalences.

$$\begin{array}{ccc}
 GM_{\mathbb{Q}} & \xrightleftharpoons[\Pi]{\Delta} & \prod_{i \in I} L_{NE_i} GM_{\mathbb{Q}} \\
 \uparrow \mathbb{N} \downarrow \mathbb{N}^{\#} & & \uparrow \prod_{i \in I} \mathbb{N} \downarrow \prod_{i \in I} \mathbb{N}^{\#} \\
 G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^{+} & \xrightleftharpoons[\Pi]{\Delta} & \prod_{i \in I} L_{E_i} G\mathcal{J}\mathcal{S}_{\mathbb{Q}}^{+}
 \end{array}
 \qquad
 \begin{array}{ccc}
 S_{\mathbb{Q}}\text{-mod} & \xrightleftharpoons[\Pi]{\Delta} & \prod_{i \in I} L_{NE_i} S_{\mathbb{Q}}\text{-mod} \\
 \uparrow S_{\mathbb{Q}} \wedge (-) \downarrow U & & \uparrow \prod_{i \in I} S_{\mathbb{Q}} \wedge (-) \downarrow \prod_{i \in I} U \\
 GM_{\mathbb{Q}} & \xrightleftharpoons[\Pi]{\Delta} & \prod_{i \in I} L_{NE_i} GM_{\mathbb{Q}}
 \end{array}$$

Proof The left hand diagram consists of Quillen equivalences by Corollary 2.1.13 and Theorem 3.3.1. The righthand diagram consists of Quillen equivalences by Theorem 2.1.12 and Proposition 2.1.18. It is easy to see that the two squares of left adjoints above commute and hence so do the squares of right adjoints. ■

Remark 3.3.3 There is a Quillen pair $S_{\mathbb{Q}}\text{-mod} \rightleftarrows L_{NE_i} S_{\mathbb{Q}}\text{-mod}$ arising from the change of model structure functors $\text{Id} : GM_{\mathbb{Q}} \rightleftarrows L_{NE_i} GM_{\mathbb{Q}} : \text{Id}$.

3.4 Idempotent Families Induce Splittings

Equivariant considerations give us a class of examples where we can apply our splitting result: Theorem 3.4.14. In particular our splitting for the $O(2)$ case will be of this form.

Definition 3.4.1 A collection of subgroups of G , \mathcal{F} , is called a **family** if it is closed under conjugation and taking subgroups. The complement of this set in the set of all subgroups of G is a **cofamily**, the cofamily associated to the family \mathcal{F} will be denoted $\widetilde{\mathcal{F}}$.

We have the **universal \mathcal{F} -space** $E\mathcal{F}$. This is a G -CW complex constructed from cells of orbit type G/H with $H \in \mathcal{F}$. This space has the universal property: $E\mathcal{F}^H$ is a contractible space for $H \in \mathcal{F}$ and $E\mathcal{F}^H = \emptyset$ for $H \notin \mathcal{F}$. Define a map $\varepsilon : E\mathcal{F}_+ \rightarrow S^0$ by using the projection $E\mathcal{F} = E\mathcal{F}^e \rightarrow *$ and then adding a disjoint point to both. The cofibre of ε , $C\varepsilon$, will be called the **universal $\widetilde{\mathcal{F}}$ -space** and will be written $\widetilde{E\mathcal{F}}$. Applying the basic fact: $Cf^H \cong (Cf)^H$, we can then see by a simple calculation that $\widetilde{E\mathcal{F}}^H$ is a contractible space for $H \in \mathcal{F}$ and $\widetilde{E\mathcal{F}}^H = S^0$ for $H \notin \mathcal{F}$.

Definition 3.4.2 We set $\mathcal{F}\mathcal{J}\mathcal{S} = L_{E\mathcal{F}_+} G\mathcal{J}\mathcal{S}$ and $\widetilde{\mathcal{F}}\mathcal{J}\mathcal{S} = L_{\widetilde{E\mathcal{F}}} G\mathcal{J}\mathcal{S}$, these are known as the **Bousfield \mathcal{F} -model structure** and **Bousfield $\widetilde{\mathcal{F}}$ -model structure** on $G\mathcal{J}\mathcal{S}$ ([MM02, Chapter IV, Section 6]). We let $[X, Y]^{\mathcal{F}}$ denote the set of maps between spectra X and Y in the homotopy category of $\mathcal{F}\mathcal{J}\mathcal{S}$ and similarly we use $[X, Y]^{\widetilde{\mathcal{F}}}$ for $\widetilde{\mathcal{F}}\mathcal{J}\mathcal{S}$. In turn we have rationalised categories $\mathcal{F}\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ and $\widetilde{\mathcal{F}}\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ which are $\mathcal{F}\mathcal{J}\mathcal{S}$ and $\widetilde{\mathcal{F}}\mathcal{J}\mathcal{S}$ localised at $S^0\mathbb{Q}$.

These are cofibrantly generated, left proper, symmetric monoidal model categories satisfying the monoid axiom by Corollary 2.1.17. We have versions for GM , where we

set $\mathcal{FM} = L_{E\mathcal{F}_+}GM$ and $\widetilde{\mathcal{FM}} = L_{E\widetilde{\mathcal{F}}}GM$, we also have the rationalised versions $\mathcal{FM}_{\mathbb{Q}}$ and $\widetilde{\mathcal{FM}}_{\mathbb{Q}}$.

Proposition 3.4.3 *The following conditions on a map $f: X \rightarrow Y$ are equivalent.*

- (i). f is an $E\mathcal{F}_+$ -equivalence.
- (ii). $f_*: \pi_*^H(X) \rightarrow \pi_*^H(Y)$ is an isomorphism for all $H \in \mathcal{F}$.

Proof This result is [MM02, Chapter IV, Proposition 6.7]. ■

A map satisfying the second condition is called an \mathcal{F} -**equivalence**. We will only need the next few results for GM , so we state them in that notation. Recall that the generating cofibrations and acyclic cofibrations of GM are defined in terms of the objects $\Sigma_V^\infty(G/H_+ \wedge S^n)$ for H a subgroup of G and V an indexing space. If we restrict these sets to only use those H in some family \mathcal{F} , then we obtain the notions of \mathcal{F} -cofibrations, \mathcal{F} -fibrations and \mathcal{F} -equivalences. These collections of maps form a model category by the following result.

Theorem 3.4.4 *The category GM has an \mathcal{F} -model structure with weak equivalences the \mathcal{F} -equivalences, cofibrations the \mathcal{F} -cofibrations and fibrations as defined by the lifting property. This is a compactly generated proper model structure and the identity functor gives the left adjoint of a Quillen equivalence from the \mathcal{F} -model structure on GM to \mathcal{FM} .*

Proof This is [MM02, Chapter IV Theorems 6.5 and 6.9]. ■

Let N be a normal subgroup of G , then the subgroups of N form a family $\mathcal{F}(N)$ of subgroups of G . We denote the $\mathcal{F}(N)$ -model structure on GM by $GM(N)$. This model structure coincides with the model structure on GM lifted (see Lemma 7.3.6) over the right adjoint $\iota_N^*: GM_U \rightarrow NM_{\iota_N^*U}$ (where U is a G -universe, so ι_N^*U is an N -universe). Hence there is a Quillen pair $G_+ \wedge_N (-) : NM \rightleftarrows GM(N) : \iota_N^*$.

Proposition 3.4.5 *For N a normal subgroup of G , $GM(N)$ is a monoidal model category that satisfies the monoid axiom.*

Proof The identity map $GM(N) \rightarrow GM$ is a left Quillen functor, since the generating cofibrations and acyclic cofibrations of $GM(N)$ are a subset of those for GM . Hence the pushout product and monoid axioms follow from those for GM . ■

Theorem 3.4.6 *For any orthogonal spectra X and Y we have natural isomorphisms*

$$\begin{aligned} [X, Y]^{\mathcal{F}} &\cong [X \wedge E\mathcal{F}_+, Y \wedge E\mathcal{F}_+]^G \\ [X, Y]^{\widetilde{\mathcal{F}}} &\cong [X \wedge E\widetilde{\mathcal{F}}, Y \wedge E\widetilde{\mathcal{F}}]^G. \end{aligned}$$

Thus we have an equivalence of categories between $\mathrm{Ho} \widetilde{\mathcal{F}}\mathcal{IS}$ and the full subcategory of objects $X \wedge E\mathcal{F}_+$ in $\mathrm{Ho} G\mathcal{IS}$. Equally there is an equivalence of categories between $\mathrm{Ho} \widetilde{\mathcal{F}}\mathcal{IS}$ and the full subcategory of objects $X \wedge E\widetilde{\mathcal{F}}$ in $\mathrm{Ho} G\mathcal{IS}$. The map $\rho: X \rightarrow F(E\mathcal{F}_+, X)$ induced by $E\mathcal{F}_+ \rightarrow S^0$ gives an $E\mathcal{F}_+$ -equivalence from X to an object that is $E\mathcal{F}_+$ -local.

Proof This is [MM02, Chapter IV, Theorems 6.11 and 6.14] and the last sentence is [MM02, Chapter IV, Proposition 6.4]. ■

Lemma 3.4.7 *For any family \mathcal{F} , $\pi_*^H(E\mathcal{F}_+ \wedge E\widetilde{\mathcal{F}}) = 0$.*

Proof We apply the functors Φ^H and the result follows from Proposition 3.1.10. ■

Corollary 3.4.8 *Take a family of subgroups \mathcal{F} , then a map $f: X \rightarrow Y$ is a π_* -isomorphism if and only if f is an $E\mathcal{F}_+$ -equivalence and an $E\widetilde{\mathcal{F}}$ -equivalence.*

Proof We have the cofibre sequence $E\mathcal{F}_+ \rightarrow \mathbb{S} \rightarrow E\widetilde{\mathcal{F}}$ and an \mathbb{S} -equivalence is a π_* -isomorphism. ■

Definition 3.4.9 Let \mathcal{F} be a family of subgroups of G such that \mathcal{F} is an open and closed G -invariant subspace of $S_f G$ that is a union of \sim classes. Then we call such a collection an **idempotent family**.

An idempotent family \mathcal{F} of G corresponds via tom Dieck's isomorphism to an idempotent $e_{\mathcal{F}}$ of the rational Burnside ring of G and the associated cofamily corresponds to the complement $1 - e_{\mathcal{F}}$, which we will also denote by $e_{\widetilde{\mathcal{F}}}$.

Lemma 3.4.10 *Consider a short exact sequence of compact Lie groups $1 \rightarrow G_e \rightarrow G \rightarrow F \rightarrow 1$. Then \mathcal{F} , the set of subgroups of G_e (the identity component of G), is an idempotent family.*

Proof The identity component of G is normal, hence the set of subgroups of G_e is closed under conjugation and taking subgroups, thus \mathcal{F} is a family. Recall that since G is compact, $F = G/G_e$ is finite. Let \mathcal{F}' be the set of conjugacy classes of groups in \mathcal{F} with finite index in their normaliser. Take $H \in \mathcal{F}'$, by [Bre72, Chapter II, Corollary 5.6] we know that if $K \in \mathcal{F}G$ is in some sufficiently small neighbourhood of H in the space $\mathcal{F}G$, then K is subconjugate to H and so K is a subgroup of G_e . It follows that \mathcal{F}' is open in $\mathcal{F}G/G$. Now take (K) to be in $(\mathcal{F}G/G) \setminus \mathcal{F}'$, so there is a $g \in G \setminus G_e$ such that $K \cap gG_e$ is non-empty. Then any $L \in \mathcal{F}G$ that is sufficiently close to K also has a non-trivial intersection with gG_e so L is not a subgroup of G_e , it follows that \mathcal{F}' is also closed. Hence $e_{\mathcal{F}}$, the characteristic function of \mathcal{F}' , is a continuous map $\mathcal{F}G/G \rightarrow \mathbb{Q}$. Thus $e_{\mathcal{F}}$ is an idempotent, since $e_{\mathcal{F}}(H) = 1$ if $(H) \in \mathcal{F}$ and zero otherwise. It follows that the support of $e_{\mathcal{F}}$ (a subset of $S_f G$) is \mathcal{F} . ■

Lemma 3.4.11 *Let \mathcal{F} be an idempotent family of subgroups of G , with corresponding idempotent $e_{\mathcal{F}}$. Then the composite $E\mathcal{F}_+ \rightarrow S^0 \rightarrow e_{\mathcal{F}}S^0$ is a rational π_* -isomorphism of orthogonal G -spectra. Equally $E\widetilde{\mathcal{F}}$ and $e_{\widetilde{\mathcal{F}}}S^0$ are rationally equivalent. Furthermore a map f is an $E\mathcal{F}_+$ -equivalence if and only if $\iota_H^*(e_{\mathcal{F}})\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism. A map f is an $E\widetilde{\mathcal{F}}$ -equivalence if and only if $\iota_H^*(e_{\widetilde{\mathcal{F}}})\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism.*

Proof Look at the geometric fixed points of $E\mathcal{F}_+ \rightarrow S^0 \rightarrow e_{\mathcal{F}}S^0$ to see that this is a rational equivalence. Similarly we have a zig-zag of rational equivalences $e_{\widetilde{\mathcal{F}}}S^0 \rightarrow e_{\widetilde{\mathcal{F}}}E\widetilde{\mathcal{F}} \leftarrow E\widetilde{\mathcal{F}}$. The second statement then follows by Corollary 3.1.7. ■

Corollary 3.4.12 *For an idempotent family \mathcal{F} and idempotents as above we have the following collection of rational equivalences for any orthogonal G -spectrum X .*

$$\begin{aligned} e_{\mathcal{F}}X &\simeq e_{\mathcal{F}}\mathbb{S} \wedge X \simeq E\mathcal{F}_+ \wedge X \\ e_{\widetilde{\mathcal{F}}}X &\simeq e_{\widetilde{\mathcal{F}}}\mathbb{S} \wedge X \simeq E\widetilde{\mathcal{F}} \wedge X \end{aligned}$$

Lemma 3.4.13 *For \mathcal{F} an idempotent family, the categories $\mathcal{F}\mathcal{I}\mathcal{S}_{\mathbb{Q}}$, $\widetilde{\mathcal{F}}\mathcal{I}\mathcal{S}_{\mathbb{Q}}$ (and their S -module counterparts $\mathcal{F}\mathcal{M}_{\mathbb{Q}}$ and $\widetilde{\mathcal{F}}\mathcal{M}_{\mathbb{Q}}$) are right proper.*

Proof Let $e \in A(G) \otimes \mathbb{Q}$ be an idempotent, then for any exact sequence of $A(G) \otimes \mathbb{Q}$ -modules $\cdots \rightarrow M_i \rightarrow M_{i-1} \rightarrow \cdots$ the sequence $\cdots \rightarrow eM_i \rightarrow eM_{i-1} \rightarrow \cdots$ is exact. Right properness follows from the proof of Lemma 2.2.3 by applying $e_{\mathcal{F}}$ (or $e_{\widetilde{\mathcal{F}}}$) to the long exact sequence of rational homotopy groups of a fibration. ■

Theorem 3.4.14 *For G a compact Lie group and \mathcal{F} an idempotent family of subgroups of G , we have a strong monoidal Quillen equivalence of cofibrantly generated, proper, monoidal model categories satisfying the monoid axiom*

$$\Delta : G\mathcal{I}\mathcal{S}_{\mathbb{Q}} \rightleftarrows \mathcal{F}\mathcal{I}\mathcal{S}_{\mathbb{Q}} \times \widetilde{\mathcal{F}}\mathcal{I}\mathcal{S}_{\mathbb{Q}} : \prod.$$

In particular we have the following natural isomorphism for any G -spectra X and Y

$$[X, Y]_{\mathbb{Q}}^G \cong [X \wedge E\mathcal{F}_+, Y \wedge E\mathcal{F}_+]_{\mathbb{Q}}^G \oplus [X \wedge E\widetilde{\mathcal{F}}, Y \wedge E\widetilde{\mathcal{F}}]_{\mathbb{Q}}^G.$$

Proof By Lemma 3.4.7 and Proposition 3.1.8 we see that $E\mathcal{F}_+$ and $E\widetilde{\mathcal{F}}$ satisfy the assumptions of Theorem 3.2.4. The description of $[X, Y]_{\mathbb{Q}}^G$ follows from Theorem 3.4.6 and Theorem 2.2.4. ■

Thus in terms of rational cohomology theories we have a natural isomorphism $E^* \cong E_{\mathcal{F}}^* \oplus E_{\widetilde{\mathcal{F}}}^*$ for $E_{\mathcal{F}}$ an $E\mathcal{F}_+$ -localisation of E and $E_{\widetilde{\mathcal{F}}}$ an $E\widetilde{\mathcal{F}}$ -localisation of E . The proof shows that $E_{\mathcal{F}} \vee E_{\widetilde{\mathcal{F}}}$ is rationally weakly equivalent to E and hence in terms of rational homology theories $E_* \cong (E_{\mathcal{F}})_* \oplus (E_{\widetilde{\mathcal{F}}})_*$.

Remark 3.4.15 Consider $G\mathcal{I}\mathcal{S}_{\mathbb{Q}}$ and fix a family \mathcal{F} , we still have $E\mathcal{F}_+ \rightarrow S^0 \rightarrow E\widetilde{\mathcal{F}}$ and the smash product of these two spectra is trivial. When will $E\mathcal{F}_+ \vee E\widetilde{\mathcal{F}}$ and \mathbb{S} be rationally equivalent? As described in Corollary 3.2.5, if there is a rational equivalence then there must be an idempotent $e \in A(G) \otimes \mathbb{Q}$ such that $E\mathcal{F}_+$ is rationally equivalent to $e\mathbb{S}$, whence \mathcal{F} will correspond to an idempotent of $A(G) \otimes \mathbb{Q}$. Conversely if \mathcal{F} corresponds to an idempotent of $A(G) \otimes \mathbb{Q}$ then $E\mathcal{F}_+ \vee E\widetilde{\mathcal{F}}$ and \mathbb{S} will be rationally equivalent. So we only obtain these splittings when \mathcal{F} is an idempotent family.

Part II

Finite Groups

Chapter 4

Rational G -Spectra for Finite G

We reprove the result of [SS03b, Example 5.1.2] using the methods of [GS] to classify rational G -spectra in terms of an algebraic model for finite G (see Corollary 4.3.12). We describe the algebraic model in the first section and apply our splitting result to the category of rational G -spectra in the second. In the third we compare each split piece to the relevant part of the algebraic model. Our input to this new proof consists of three pieces of work: the splitting of the category of G -spectra, showing that the results of [GS] can be applied to this setting and proving Proposition 4.3.8 and Theorem 4.3.9 to complete the classification. Section 4.3 is the first time we will need to consider right modules over an enriched category and use the Morita equivalence of Theorem 5.4.3. Thus, we also include a chapter on enriched categories in this part of the thesis.

4.1 The Algebraic Category

We use the description of rational G -cohomology theories as implied by [GM95, Appendix A] to obtain the following definition. Since the algebraic model is a product of categories, we work piecewise and replace $dg\mathbb{Q}W_G H\text{-mod}$ by $\text{mod-}\mathcal{E}_a^H$ in Proposition 4.1.8.

Definition 4.1.1 The algebraic model for rational G -spectra for finite G is

$$dg\mathcal{A}(G) = \prod_{(H) \leq G} dg\mathbb{Q}W_G H\text{-mod}.$$

Here we are using the projective model structure on the categories of modules.

Remark 4.1.2 The rational group ring of G is a Hopf algebra with co-commutative coproduct. Since $\mathbb{Q}G \otimes_{\mathbb{Q}} \mathbb{Q}G \cong \mathbb{Q}(G \times G)$ the product is induced by the group multiplication $G \times G \rightarrow G$, the coproduct by the diagonal $G \rightarrow G \times G$ and the antipode by the inversion map $G \rightarrow G$. Because $\mathbb{Q}G$ is a Hopf-algebra there is a monoidal product on $dg\mathbb{Q}G$ -modules.

Definition 4.1.3 For M and N in $dg\mathbb{Q}G\text{-mod}$ define their tensor product to be $M \otimes_{\mathbb{Q}} N$ with the diagonal G -action. There is an internal function object, defined as $\underline{\text{Hom}}_{\mathbb{Q}}(M, N)$ with G -action by conjugation.

Definition 4.1.4 For any $X \in dg\mathbb{Q}G\text{-mod}$, there is an $dg\mathbb{Q}G$ -map $Av_G: X \rightarrow X^G$ defined by $Av_G(x) = |G|^{-1} \sum_{g \in G} gx$.

Lemma 4.1.5 *The tensor product above gives $dg\mathbb{Q}G\text{-mod}$ the structure of a closed symmetric monoidal $dg\mathbb{Q}$ -model category that satisfies the monoid axiom.*

Proof Let Δ be the coproduct of $\mathbb{Q}G$, T be the interchange of factors map and ν_M and ν_N be the $\mathbb{Q}G$ -action maps of $dg\mathbb{Q}G$ -modules M and N . The action of $\mathbb{Q}G$ on $M \otimes_{\mathbb{Q}} N$ is then defined as the composite:

$$\begin{array}{ccc} \mathbb{Q}G \otimes_{\mathbb{Q}} M \otimes_{\mathbb{Q}} N & \xrightarrow{\Delta} & \mathbb{Q}G \otimes_{\mathbb{Q}} \mathbb{Q}G \otimes_{\mathbb{Q}} M \otimes_{\mathbb{Q}} N \\ & \text{Id} \otimes T \otimes \text{Id} \searrow & \downarrow \\ & & \mathbb{Q}G \otimes_{\mathbb{Q}} M \otimes_{\mathbb{Q}} \mathbb{Q}G \otimes_{\mathbb{Q}} N \\ & \nu_M \otimes \nu_N \searrow & \downarrow \\ & & M \otimes_{\mathbb{Q}} N. \end{array}$$

It is clear that this is a commutative, associative monoidal product with unit \mathbb{Q} (with trivial G -action). To prove that the pushout product axiom holds, it suffices (by [SS00, Lemma 3.5(1)]) to check the following pair of conditions.

- (i). If f and g are generating cofibrations then the pushout product, $f \square g$, is a cofibration.
- (ii). If f is a generating cofibration and g is a generating acyclic cofibration then $f \square g$ is a weak equivalence.

Let f and g be generating cofibrations, then $f \square g$ is an inclusion with cokernel $\mathbb{Q}(G \otimes G)$ (in some degree). We claim that this cokernel is cofibrant: by Remark 4.1.2 this is isomorphic (as a $\mathbb{Q}G$ -module) to $\bigoplus_{g \in G} \mathbb{Q}G$ and the claim follows from the fact that $\mathbb{Q}G$ is cofibrant as a $\mathbb{Q}G$ -module. Taking a generating cofibration f and a generating acyclic cofibration g , it follows that $f \square g$ is a weak equivalence since both the domain and codomain are acyclic.

There is a strong symmetric monoidal adjoint pair $\varepsilon^* : dg\mathbb{Q} \rightleftarrows dg\mathbb{Q}G : (-)^G$, where $\varepsilon^*(X)$ is X with trivial action and the right adjoint is the fixed point functor. We show that this is a Quillen pair by proving that the right adjoint preserves fibrations and weak equivalences. Take $f: X \rightarrow Y$ a surjection and let $y \in Y^G$, then there is an x such that $f(x) = y$. Since $Av_G(x) \in X^G$ and $f(Av_G(x)) = Av_G(f(x)) = Av_G(y) = y$, it follows that f^G is surjective. That $(-)^G$ preserves homology isomorphisms follows immediately from the isomorphism $H_*(X^G) \cong (H_*X)^G$. Thus $dg\mathbb{Q}G$ is a $dg\mathbb{Q}$ -model category.

For a commutative ring R , there is a tensor product on $dgR\text{-mod}$, $- \otimes_R -$. This category with the projective model structure satisfies the monoid axiom, as proven in [Shi07b, Proposition 3.1]. This result in the case $R = \mathbb{Q}$ will imply that $dg\mathbb{Q}G\text{-mod}$ satisfies the monoid axiom. Thus we write out the proof of [Shi07b, Proposition 3.1],

adapted to the notation of $\mathbb{Q}G$. By [SS00, Lemma 3.5(2)] it suffices to show that transfinite composition and pushouts of maps of the form $j \otimes \text{Id}_Z : A \otimes Z \rightarrow B \otimes Z$ are weak equivalences, for j a generating acyclic cofibration. The generating acyclic cofibrations for $dg\mathbb{Q}G$ are maps $0 \rightarrow D^n(\mathbb{Q}G)$ for some integer n . Take any $Z \in dg\mathbb{Q}G\text{-mod}$, then it is easy to check that $D^n(\mathbb{Q}G) \otimes_{\mathbb{Q}} Z$ is also acyclic. Then we note that $0 \rightarrow D^n(\mathbb{Q}G) \otimes_{\mathbb{Q}} Z$ is an injection and a homology isomorphism. Such maps are closed under pushouts and transfinite compositions (they are acyclic cofibrations in the injective model structure on $dg\mathbb{Q}G$ -modules), hence the monoid axiom holds. ■

Note that this proves shows that $\otimes_i \mathbb{Q}G$ is cofibrant as a $dg\mathbb{Q}G$ -module.

Lemma 4.1.6 *The model category of $dg\mathbb{Q}G\text{-mod}$ is generated by $\mathbb{Q}G$.*

Definition 4.1.7 Let $\mathcal{G}_{a,G} = \{\mathbb{Q}, \mathbb{Q}G, \mathbb{Q}(G \times G), \mathbb{Q}(G \times G \times G), \dots\}$ and let $\mathcal{E}_{a,G}$ be the $dg\mathbb{Q}$ -category with object set $\mathcal{G}_{a,G}$ and $dg\mathbb{Q}$ -mapping object given by $\mathcal{E}_{a,G}(X, Y) = \underline{\text{Hom}}_{\mathbb{Q}}(X, Y)^G$. Now we define $\mathcal{G}_{a,G}^H = \mathcal{G}_{a,W_G H}$ and $\mathcal{E}_{a,G}^H = \mathcal{E}_{a,W_G H}$. We will usually suppress the G and reduce this notation to \mathcal{G}_a^H and \mathcal{E}_a^H .

Proposition 4.1.8 *There is a strong symmetric monoidal Quillen equivalence*

$$(-) \otimes_{\mathcal{E}_a^H} \mathcal{G}_a^H : \text{mod-}\mathcal{E}_a^H \rightleftarrows dg\mathbb{Q}W_G H\text{-mod} : \underline{\text{Hom}}(\mathcal{G}_a^H, -)$$

and furthermore this is an adjunction of closed symmetric monoidal $dg\mathbb{Q}$ -model categories.

Proof This is an application of [GS, Proposition 3.6], see Theorem 5.4.3. ■

Lemma 4.1.9 *There is an isomorphism of rings $\underline{\text{Hom}}_{\mathbb{Q}}(\mathbb{Q}G, \mathbb{Q}G)^G \cong \mathbb{Q}G$.*

Proof This is a standard result. A G -map $\mathbb{Q}G \rightarrow \mathbb{Q}G$ is defined by the image of 1, let \tilde{g} represent the G -map which sends $1 \rightarrow g$, for $g \in G$. These are a set of generators for $\underline{\text{Hom}}_{\mathbb{Q}}(\mathbb{Q}G, \mathbb{Q}G)^G$. We define the above ring isomorphism to be that map which sends \tilde{g} to $g^{-1} \in G$. ■

4.2 Topological Models and Splitting

We take tom-Dieck's isomorphism in the case of a finite group, $A(G) \otimes \mathbb{Q} \cong \prod_{(K) \leqslant G} \mathbb{Q}$, and see how it applies to the model category of rational G -spectra. For each conjugacy class of subgroups, $(H) \leqslant G$, there is an idempotent e_H given by projection onto factor (H) . We use this in Theorem 4.2.4 to split the category of rational G -spectra into a collection of model categories each generated by a single object. We also provide a version of this splitting in terms of modules over a ring spectrum (Proposition 4.2.12). The advantage of this second description is that every object of the category is fibrant, which is needed for Theorem 4.3.2 (see Remark 5.4.4).

Definition 4.2.1 For a group G , with subgroups H and K , we say that K is **subconjugate** to H if the G -conjugacy class of K contains a subgroup of H , we write $K \leqslant_G H$. In turn K is **strictly subconjugate** to H if the G -conjugacy class of K contains a strict subgroup of H , the notation for this is $K <_G H$.

Definition 4.2.2 Take H a subgroup of G , then we have a pair of families of subgroups of G : $[\leq_G H]$ – the family of all subgroups of G which are subconjugate to H and $[<_G H]$ – the family of all subgroups of G which are strictly subconjugate to H . We can then form G -CW complexes $E[\leq_G H]_+$ and $E[<_G H]_+$. There is a map $E[<_G H]_+ \rightarrow E[\leq_G H]_+$, we call the cofibre of this map $E\langle H \rangle$.

Note that since $E[<_G H]_+$ and $E[\leq_G H]_+$ are cofibrant as G -spaces, the space $E\langle H \rangle$ is also cofibrant as a G -space. We can also describe $E\langle H \rangle$ as $E[\leq_G H]_+ \wedge \widetilde{E[<_G H]}$. Since geometric fixed point functors preserve cofibre sequences, $\Phi^K(E\langle H \rangle)$ is contractible unless $(K) = (H)$, whence it is non-equivariantly equivalent to \mathbb{S} .

Lemma 4.2.3 *The families $[\leq_G H]$ and $[<_G H]$ are idempotent families (see Definition 3.4.9), with corresponding idempotents $e_{[\leq_G H]} = \Sigma_{(K) \leq H} e_K$ and $e_{[<_G H]} = \Sigma_{(K) < H} e_K$.*

Proof Since G is finite, any collection of conjugacy classes of subgroups of G defines a unique idempotent of the Burnside ring. ■

Theorem 4.2.4 *There is a strong symmetric monoidal Quillen equivalence between the category of rational G -spectra and the product of the categories $L_{E\langle H \rangle} GM_{\mathbb{Q}}$, as H runs over the set of conjugacy classes of subgroups of G .*

$$\Delta : GM_{\mathbb{Q}} \xleftrightarrow{\quad} \prod_{(H) \leq G} L_{E\langle H \rangle} GM_{\mathbb{Q}} : \prod$$

The left adjoint takes a rational G -spectrum X , to the constant collection of X in every factor. The right adjoint takes the collection Y_H to the G -spectrum $\prod_{(H)} Y_H$.

Proof Using Theorems 3.2.4 and 3.3.2 all we have to show is that $E\langle H \rangle \wedge E\langle K \rangle$ is rationally acyclic whenever $(H) \neq (K)$ and $\bigvee_{(H) \leq G} E\langle H \rangle$ is rationally equivalent to \mathbb{S} (working temporarily in orthogonal spectra). We claim that $E\langle H \rangle$ is rationally equivalent to $e_H \mathbb{S}$, from which both conditions above follow immediately. Since $E\langle H \rangle$ can be constructed as $E[\leq_G H]_+ \wedge \widetilde{E[<_G H]}$ it is rationally equivalent to $e_{[\leq_G H]} \mathbb{S} \wedge (1 - e_{[<_G H]}) \mathbb{S}$ by Lemma 3.4.11 and hence rationally equivalent to $e_{[\leq_G H]} (1 - e_{[<_G H]}) \mathbb{S}$. We then have a zig-zag of $\pi_*^{\mathbb{Q}}$ -isomorphisms

$$e_H \mathbb{S} \rightarrow e_H (e_{[\leq_G H]} (1 - e_{[<_G H]}) \mathbb{S}) \leftarrow e_{[\leq_G H]} \mathbb{S} \wedge (1 - e_{[<_G H]}) \mathbb{S}$$

which proves our claim. Since $\mathbb{S} \rightarrow \bigvee_{(H) \leq G} e_H \mathbb{S}$ is a rational equivalence we have our result. ■

Corollary 4.2.5 *A map $f : X \rightarrow Y$ in $GM_{\mathbb{Q}}$ is a rational $E\langle H \rangle$ -equivalence if and only if $e_H f : e_H X \rightarrow e_H Y$ is a rational equivalence.*

Proof One checks this by looking at geometric fixed points. ■

Corollary 4.2.6 *The Quillen equivalence*

$$\Delta : GM_{\mathbb{Q}} \xleftrightarrow{\quad} \prod_{(H) \leq G} L_{E\langle H \rangle} GM_{\mathbb{Q}} : \prod$$

is a symmetric monoidal adjunction of closed Sp_+^Σ -algebras.

We now fix some subgroup H and study $L_{E\langle H \rangle}GM_{\mathbb{Q}}$.

Lemma 4.2.7 *There is an equality of model structures:*

$$L_{E\langle H \rangle}GM_{\mathbb{Q}} = L_{E\langle H \rangle}L_{E[\leq_G H]_+}GM_{\mathbb{Q}}$$

that is to say, the weak equivalences, cofibrations and fibrations agree.

Proof We claim that we have the equality: $L_{E\langle H \rangle}GM_{\mathbb{Q}} = L_{E\langle H \rangle}L_{E[\leq_G H]_+}GM_{\mathbb{Q}}$, where the cofibrations agree by definition. The map $E[\leq_G H] \rightarrow *$ is an $E\langle H \rangle$ equivalence (look at the homotopy groups and the idempotents). Hence, considering the cofibre sequence which defines $E[\leq_G H]$ we have a weak equivalence.

$$E[\leq_G H]_+ \wedge E\langle H \rangle \rightarrow E\langle H \rangle$$

It then follows that an $E[\leq_G H]_+ \wedge E\langle H \rangle$ -equivalence is an $E\langle H \rangle$ -equivalence. So the weak equivalences of $L_{E\langle H \rangle}GM_{\mathbb{Q}}$ and $L_{E\langle H \rangle}L_{E[\leq_G H]_+}GM_{\mathbb{Q}}$ agree and thus we have proved our claim. The result then follows immediately. ■

Remark 4.2.8 The weak equivalences of $L_{E[\leq_G H]_+}GM_{\mathbb{Q}}$ are those maps f such that $\pi_*^K(f) \otimes \mathbb{Q}$ is an isomorphism for all $K \leq_G H$. This is [MM02, IV, Proposition 6.7].

Lemma 4.2.9 *A map f in $L_{E\langle H \rangle}GM_{\mathbb{Q}}$ is a weak equivalence if and only if the induced map of homotopy groups $i_H^*(e_H)\pi_*^H(f) \otimes \mathbb{Q}$ is a isomorphism.*

Proof Lemma 4.2.7 and Corollary 4.2.5 show that f is a weak equivalence if and only if $i_K^*(e_H)\pi_*^K(f) \otimes \mathbb{Q}$ is an isomorphism for all $K \leq_G H$. Now note that K is a strict subset of H then $i_K^*(e_H) = 0$, hence for any map f , $i_K^*(e_H)\pi_*^K(f) \otimes \mathbb{Q}$ will be an isomorphism. ■

Let $H \leq G$ then there is an idempotent $e_H^G \in A(G) \otimes \mathbb{Q}$ and $\iota_H^*(e_H^G) = e_H^H \in A(H) \otimes \mathbb{Q}$. Thus we reword the lemma above as: a map f in $L_{E\langle H \rangle}GM_{\mathbb{Q}}$ is a weak equivalence if and only if $e_H^H\pi_*^H(f)$ is a isomorphism.

Now we must obtain a version of this splitting with every object of the split categories fibrant (see Remark 5.4.4). To achieve this we will apply the same approach as in Section 1.5 and construct a suitable ring spectrum via Theorem 1.5.7.

Lemma 4.2.10 *Given $\mathcal{F} \subset \mathcal{F}'$, families of subgroups of G , the cofibre of the induced map of classifying spaces $E\mathcal{F}_+ \rightarrow E\mathcal{F}'_+$, is a cell complex.*

Proof One can either prove this directly or note that it follows by the same proof as for Lemma 1.5.6. ■

Recall from [EKMM97, Chapter III, Proposition 2.6] that the smash product of a pair of cell complexes is also a cell complex.

Lemma 4.2.11 *There is an $S_{\mathcal{M}}^0\mathbb{Q} \wedge E\langle H \rangle$ -local commutative cell S -algebra S_H whose unit map is a rational $E\langle H \rangle$ -equivalence and an inclusion of cell complexes. Furthermore every S_H -module is $S_{\mathcal{M}}^0\mathbb{Q} \wedge E\langle H \rangle$ -local.*

Proof This result is an application of Theorem 1.5.7 using the cell object $S_{\mathcal{M}}^0\mathbb{Q} \wedge E\langle H \rangle$ to create a commutative ring S_H which is the $S_{\mathcal{M}}^0\mathbb{Q} \wedge E\langle H \rangle$ -localisation of S . By construction the unit map $S \rightarrow S_H$ is a rational $E\langle H \rangle$ -equivalence, hence $S_{\mathcal{M}}^0\mathbb{Q} \wedge E\langle H \rangle$ is π_* -isomorphic to $S_H \wedge S_{\mathcal{M}}^0\mathbb{Q} \wedge E\langle H \rangle$. Since S_H is $S_{\mathcal{M}}^0\mathbb{Q}$ -local, it has rational homotopy groups, thus there is a zig-zag of weak equivalences $S_{\mathcal{M}}^0\mathbb{Q} \wedge S_H \leftarrow \widehat{c}S \wedge S_H \rightarrow S_H$. Equally S_H is weakly equivalent to $S_H \wedge \bigvee_{(K)} E\langle K \rangle$. Since S_H is $E\langle H \rangle$ -local, $S_H \wedge E\langle K \rangle$ is acyclic whenever $(H) \neq (K)$ (this is part of the proof of Theorem 3.2.4). It follows that $S_H \wedge \bigvee_{(K)} E\langle K \rangle$ is weakly equivalent to $S_H \wedge E\langle H \rangle$. Thus S_H is π_* -isomorphic to $S_{\mathcal{M}}^0\mathbb{Q} \wedge E\langle H \rangle$. Now we can use the proofs of Lemmas 2.3.2 and 2.3.3 to show the last statement. ■

Proposition 4.2.12 *The adjoint pair of the free S_H -module functor and the forgetful functor*

$$S_H \wedge (-) : L_{E\langle H \rangle} G\mathcal{M}_{\mathbb{Q}} \rightleftarrows S_H\text{-mod} : U$$

is a strong symmetric monoidal Quillen equivalence.

Proof The proof of Theorem 2.3.4 can be applied in this case. The two points to note are: an $S^0\mathbb{Q} \wedge E\langle H \rangle$ -equivalence between $S^0\mathbb{Q} \wedge E\langle H \rangle$ -local objects is a π_* -isomorphism and the unit map is an $S^0\mathbb{Q} \wedge E\langle H \rangle$ -equivalence. ■

Lemma 4.2.13 *The object $\widehat{c}(G/H_+) \wedge S_H$ is a G -compact, cofibrant and fibrant generator of $S_H\text{-mod}$.*

Proof Every object of $S_H\text{-mod}$ is fibrant and since $\widehat{c}(G/H_+)$ is a cofibrant spectrum, so is $\widehat{c}(G/H_+) \wedge S_H$. This object is G -compact since the right adjoint U commutes with filtered colimits and $\widehat{c}(G/H_+)$ is a G -compact G -spectrum. Since the weak equivalences of $L_{E\langle H \rangle} G\mathcal{M}_{\mathbb{Q}}$ are the $i_H^*(e_H)\pi_*^H(f) \otimes \mathbb{Q}$ -isomorphisms it follows that $\widehat{c}(G/H_+)$ generates this model category. Hence $\widehat{c}(G/H_+) \wedge S_H$ generates $S_H\text{-mod}$. ■

4.3 Comparing Ringoids

We use the results of [GS] to replace $S_H\text{-mod}$ by $\text{mod-}\mathcal{E}_{top}^H$ (Theorem 4.3.2). This category is Quillen equivalent to the category $\text{mod-}\mathcal{E}_t^H$ (Theorem 4.3.3). We show that the homology of \mathcal{E}_t^H is given by \mathcal{E}_a^H in Proposition 4.3.8. Then we use Theorem 4.3.9 to prove that \mathcal{E}_t^H and \mathcal{E}_a^H are quasi-isomorphic. This will complete our classification of rational G -spectra for finite G and we summarise this classification in Corollary 4.3.12.

Definition 4.3.1 Let \mathcal{G}_{top}^H be the set of all smash products of $\widehat{c}(G/H_+) \wedge S_H$ (including the identity as the zero-fold smash). Let \mathcal{E}_{top}^H be the spectral category on the objects of \mathcal{G}_{top}^H , so by the proof of Theorem 8.2.6,

$$\mathcal{E}_{top}^H(X, Y) = \text{Sing } \mathbb{U}(i^* \mathbb{N}^{\#} U F_{S_H}(X, Y))^G.$$

With the exception of the unit, all objects of \mathcal{G}_{top}^H are cofibrant and all objects are fibrant. We use the results of [GS] to replace this category of S_H -modules by a category of modules over an endomorphism ringoid \mathcal{E}_{top}^H .

Theorem 4.3.2 *The adjoint pair*

$$(-) \wedge_{\mathcal{E}_{top}^H} \mathcal{G}_{top}^H : \text{mod-}\mathcal{E}_{top}^H \xleftrightarrow{\quad} S_H\text{-mod} : \underline{\text{Hom}}(\mathcal{G}_{top}^H, -)$$

is a Quillen equivalence and an strong symmetric monoidal adjunction of closed symmetric monoidal spectral model categories.

Proof This follows from Theorem 5.4.3 with the adjustments as made in the proof of Theorem 9.1.2. \blacksquare

Theorem 4.3.3 *There is a zig-zag of Quillen equivalences between $\text{mod-}\mathcal{E}_{top}^H$ (enriched over Sp_+^Σ) and a category $\text{mod-}\mathcal{E}_t^H$ (enriched over $dg\mathbb{Q}\text{-mod}$). These equivalences are Quillen modules over the appropriate enrichments. This zig-zag induces an isomorphism of graded \mathbb{Q} -categories: $\pi_*(\mathcal{E}_{top}^H) \cong H_*\mathcal{E}_t^H$.*

Proof This is contained in the proof of [GS, Theorem 4.1] which is based on [Shi07b, Corollary 2.16] and we go through this in some detail in Section 9.3. \blacksquare

Remark 4.3.4 We consider the above theorem in the case of the trivial group where our work reduces to that of [Shi07b]. Here \mathcal{G}_{top} has just one object and $\text{mod-}\mathcal{E}_{top}$ is equivalent to $S_{\mathbb{Q}}\text{-mod}$. Moving from $\text{mod-}\mathcal{E}_{top}$ to $\text{mod-}\mathcal{E}_t$ is then just applying the functors of [Shi07b] to the spectrum $S_{\mathbb{Q}}$. The resulting chain complex is then weakly equivalent to \mathbb{Q} , as the comparison between $\text{mod-}\mathcal{E}_t$ and $\text{mod-}\mathcal{E}_a$ below will prove. With reference to Remark 9.3.6 this classification can be made symmetric monoidal. This is shown by using the four step comparison of [Shi07a], where we use the fibrant replacement functor of commutative rings in $Sp^\Sigma(dg\mathbb{Q}\text{-mod})$ as constructed in [Shi07a, Proposition 3]. This fibrant replacement functor comes from a model structure where weak equivalences and fibrations are defined in terms of the underlying category $Sp^\Sigma(dg\mathbb{Q}\text{-mod})$.

Proposition 4.3.5 *There is an isomorphism of rings*

$$\pi_*(F_{S_H}(G/H_+ \wedge S_H, G/H_+ \wedge S_H)^G) \cong \mathbb{Q}W_G H.$$

Proof We can make the following identifications:

$$\begin{aligned} F_{S_H}(G/H_+ \wedge S_H, G/H_+ \wedge S_H)^G &\cong F(G/H_+, G/H_+ \wedge S_H)^G \\ &\cong (G/H_+ \wedge S_H)^H. \end{aligned}$$

Thus we must calculate $\pi_*((G/H_+ \wedge S_H)^H)$, as a rational vector space. This is isomorphic to $\pi_*((G/H_+ \wedge E\langle H \rangle)^H) \otimes \mathbb{Q}$. Now $\iota_H^* E[\leq_G H]$ is H -equivariantly weakly equivalent to S , so $E\langle H \rangle$ is H -equivariantly weakly equivalent to $E\widetilde{\mathcal{F}}_H$ (see the proof of Proposition 3.1.10). Thus we have an isomorphism

$$\pi_*((G/H_+ \wedge E\langle H \rangle)^H) \otimes \mathbb{Q} \cong \pi_*(\Phi^H G/H_+) \otimes \mathbb{Q}.$$

The following is standard: $\Phi^H \Sigma^\infty G/H_+ \simeq \Sigma^\infty(G/H^H) = \Sigma^\infty W_G H$, the suspension spectrum of a finite set. Thus $\pi_*(\Phi^H G/H_+) \otimes \mathbb{Q} \cong \pi_*(W_G H_+) \otimes \mathbb{Q} \cong \mathbb{Q} W_G H$, hence there is an isomorphism of rational vector spaces $\pi_*(F_{S_H}(G/H_+ \wedge S_H, G/H_+ \wedge S_H)^G) \cong \mathbb{Q} W_G H$.

Now we prove that we have an isomorphism of rings. For each $gH \in W_G H$ there is a G -map $\widetilde{gH}: \widetilde{G/H_+} \rightarrow G/H_+$ which takes $kH \rightarrow kgH$. For $g_1 H$ and $g_2 H$ in $W_G H$, $\widetilde{g_2 H} \circ \widetilde{g_1 H} = \widetilde{g_1 g_2 H}$. The set of \widetilde{gH} for $gH \in W_G H$ generate the ring $\pi_*(F_{S_H}(G/H_+ \wedge S_H, G/H_+ \wedge S_H)^G)$. We send \widetilde{gH} to $g^{-1}H$ to obtain a ring isomorphism as desired. ■

Lemma 4.3.6 *For an integer $i \geq 1$, the i -fold product of G/H contains $|W_G H|^{i-1}$ disjoint copies of G/H . More precisely, in the Burnside ring $A(G)$*

$$G/H^{\times i} = |W_G H|^{i-1} \cdot G/H + R$$

where the remainder R consists of coset spaces G/K with $(K) \neq (H)$. Equally, there is an isomorphism of $\mathbb{Q} W_G H$ -modules, $\mathbb{Q}(W_G H^{\times i}) \cong \bigoplus_{|W_G H|^{i-1}} \mathbb{Q} W_G H$.

Proof The G -set $G/H^{\times i}$ can only consist of homogenous spaces G/K for K sub-conjugate to H . Thus, to find the number of copies of G/H in $G/H^{\times i}$ it suffices to calculate the size of the H -fixed point set: $|(G/H^{\times i})^H| = |(G/H)^H|^i$. Since $(G/H)^H = W_G H$, $|(G/H^{\times i})^H| = |W_G H|^{i-1} |(G/H)^H|$ and the result follows. The statement about $\mathbb{Q} W_G H$ -modules is obvious. ■

Proposition 4.3.7 *For integers $i, j \geq 1$,*

$$\pi_*(F_{S_H}(G/H_+^{\wedge i} \wedge S_H, G/H_+^{\wedge j} \wedge S_H)^G) \cong \text{Hom}_{\mathbb{Q} W_G H}(\mathbb{Q}(W_G H^{\times i}), \mathbb{Q}(W_G H^{\times j})).$$

Proof Using our understanding of the Burnside ring, we can write the above term as

$$\pi_*(F(R_+, G/H_+^{\wedge j} \wedge S_H)^G) \oplus \left(\bigoplus_{|W_G H|^{i-1}} \pi_*(F(G/H_+, G/H_+^{\wedge j} \wedge S_H)^G) \right)$$

where R is some wedge of spaces of form G/K_+ for $(K) \neq (H)$. We deal with the R -part first. Consider $\pi_*(F(G/K_+, G/H_+^{\wedge j} \wedge S_H)^G)$ for $(K) \neq (H)$, by arguments in the proof of Proposition 4.3.5 this is isomorphic to $i_K^*(e_H) \pi_K^*(G/H_+^{\wedge j}) \otimes \mathbb{Q}$. Now $i_K^*(e_H) = 0$ whenever $(K) \neq (H)$, so this is zero. Since all the terms of the R -part have this form, $\pi_*(F(R_+, G/H_+^{\wedge j} \wedge S_H)^G) = 0$. It remains to calculate $e_H \pi_H^*(G/H_+^{\wedge j}) \otimes \mathbb{Q}$, which by arguments in Proposition 4.3.5 is isomorphic to $\pi_*(\Phi^H G/H_+^{\wedge j}) \otimes \mathbb{Q}$ which is of course $\pi_*(W_G H_+^{\wedge j}) \otimes \mathbb{Q}$. In turn this is isomorphic to $\mathbb{Q}(W_G H^{\times j})$ and the result follows immediately. ■

Proposition 4.3.8 *There is an isomorphism of graded \mathbb{Q} -categories*

$$\mathcal{E}_a^H \cong H_* \mathcal{E}_t^H.$$

Proof By Theorem 4.3.3 it suffices to show that there is an isomorphism $\mathcal{E}_a^H \cong \pi_*(\mathcal{E}_{top}^H)$ and we begin by proving that the object sets of these categories are isomorphic. In each case there is an object σ_1 and a unit σ_0 , such that every non-unit object is a product of copies of σ_1 . For \mathcal{E}_a^H the unit is \mathbb{Q} and σ_1 is $\mathbb{Q}W_G H$. For \mathcal{E}_{top}^H the unit is S_H and $\sigma_1 = \widehat{c}(G/H_+) \wedge S_H$. Thus we define an isomorphism $\text{Ob } \mathcal{E}_a^H \rightarrow \text{Ob } H_* \mathcal{E}_{top}^H$ by taking the i -fold product of $\mathbb{Q}W_G H$ (written σ_i) to the i -fold product of $\widehat{c}(G/H_+) \wedge S_H$. We can consider these graded \mathbb{Q} -categories to have the object set: $\{\sigma_i | i \geq 0\}$. The previous result implies that $\pi_*(\mathcal{E}_{top}^H)(\sigma_i, \sigma_j) \cong \mathcal{E}_a^H(\sigma_i, \sigma_j)$ as \mathbb{Q} -modules. We must now show that this isomorphism is compatible with the composition operation in these graded- \mathbb{Q} -categories.

We have the isomorphism $\pi_*(\mathcal{E}_{top}^H)(\sigma_i, \sigma_j) \cong [\sigma_i, \sigma_j]_*^G$ where the right hand side means graded maps in the homotopy category of S_H -modules. This isomorphism specifies the composition rule of the enriched category $\pi_*(\mathcal{E}_{top}^H)$. Our calculations above allow us to write $[\sigma_i, \sigma_j]_*^G$ as

$$\left[\bigvee_{|W_G H|^{i-1}} G/H_+ \wedge S_H, \bigvee_{|W_G H|^{j-1}} G/H \wedge S_H \right]_*^G.$$

Then we define (y, x, \widetilde{gH}) , to be that map which takes the x -factor of $\bigvee_{|W_G H|^{i-1}} G/H_+$ to the y -factor of $\bigvee_{|W_G H|^{j-1}} G/H_+$ by the rule $H \mapsto gH$. This is a rational basis for $[\sigma_i, \sigma_j]_*^G$. It is easy to check that composition behaves as follows: $(z, y, \widetilde{g_2 H}) \circ (y, x, \widetilde{g_1 H}) = (z, x, \widetilde{g_1 g_2 H})$. Now we note that $\text{Hom}_{\mathbb{Q}W_G H}(\mathbb{Q}(W_G H^{\times i}), \mathbb{Q}(W_G H^{\times j}))$ is isomorphic to

$$\text{Hom}_{\mathbb{Q}W_G H} \left(\bigoplus_{|W_G H|^{i-1}} \mathbb{Q}W_G H, \bigoplus_{|W_G H|^{j-1}} \mathbb{Q}W_G H \right)$$

and write (y, x, gH) for the map which takes the x -factor of $\bigoplus_{|W_G H|^{i-1}} \mathbb{Q}W_G H$ to the y -factor of $\bigoplus_{|W_G H|^{j-1}} \mathbb{Q}W_G H$ by $H \mapsto gH$. The isomorphism of the theorem is then just: $(y, x, \widetilde{gH}) \mapsto (y, x, gH)$. ■

Theorem 4.3.9 *If \mathcal{E} is a $dg\mathbb{Q}$ -category with $H_* \mathcal{E}$ concentrated in degree zero, then \mathcal{E} is quasi-isomorphic to $H_* \mathcal{E}$ as $dg\mathbb{Q}$ -categories.*

Proof We will create a $dg\mathbb{Q}$ -category $C_0 \mathcal{E}$ and a zig-zag of quasi-isomorphisms: $\mathcal{E} \xleftarrow{\sim} C_0 \mathcal{E} \xrightarrow{\sim} H_0 \mathcal{E} = H_* \mathcal{E}$. Let C_0 be the (-1) -connective cover functor, which is right adjoint to the inclusion of $dg\mathbb{Q}\text{-mod}_+$ into $dg\mathbb{Q}\text{-mod}$. If X is a $dg\mathbb{Q}$ -module, then $(C_0 X)_n = X_n$ for $n > 0$ and $(C_0 X)_0 = \ker(\partial_0)$. We have a counit $C_0 X \rightarrow X$ in $dg\mathbb{Q}\text{-mod}$ and this is a monoidal natural transformation. Hence, given $X \wedge Y \rightarrow Z$, we have a map $C_0 X \wedge C_0 Y \rightarrow C_0 Z$ and a commuting diagram

$$\begin{array}{ccc} C_0 X \wedge C_0 Y & \longrightarrow & C_0 Z \\ \downarrow & & \downarrow \\ X \wedge Y & \longrightarrow & Z. \end{array}$$

Thus we have a $dg\mathbb{Q}$ -category $C_0 \mathcal{E}$ with a map of ringoids $C_0 \mathcal{E} \rightarrow \mathcal{E}$. Since \mathcal{E} has homology concentrated in degree zero this is a quasi-isomorphism.

For X a $dg\mathbb{Q}$ -module we have a map $C_0X \rightarrow H_0X$ which sends X_i to zero for $i > 0$ and sends $\ker(\partial_0) \rightarrow H_0X$ by the quotient. We can consider H_0 as a functor $dg\mathbb{Q}\text{-mod}_+ \rightarrow \mathbb{Q}\text{-mod}$, this has a right adjoint which includes $\mathbb{Q}\text{-mod}$ into $dg\mathbb{Q}\text{-mod}_+$ by taking a \mathbb{Q} -module M to the chain complex with M in degree zero and zeroes elsewhere. The map $C_0X \rightarrow H_0X$ is induced by the unit of this adjunction. The functor H_0 is monoidal, as is the inclusion of $\mathbb{Q}\text{-mod}$ into $dg\mathbb{Q}\text{-mod}_+$, thus we obtain a $dg\mathbb{Q}\text{-mod}_+$ -category $H_0\mathcal{E}$. Furthermore, the map $C_0X \rightarrow H_0X$ is induced by the unit of the adjunction and is a monoidal natural transformation. Thus we obtain $C_0\mathcal{E} \rightarrow H_0\mathcal{E} = H_*\mathcal{E}$, which is a quasi-isomorphism. ■

Corollary 4.3.10 *There is a zig-zag of quasi-isomorphisms of $dg\mathbb{Q}$ -categories.*

$$\mathcal{E}_t^H \xleftarrow{\sim} C_0\mathcal{E}_t^H \xrightarrow{\sim} H_*\mathcal{E}_t^H \cong \mathcal{E}_a^H$$

hence there is a zig-zag of Quillen equivalences of $dg\mathbb{Q}\text{-mod}$ -model categories.

$$\text{mod-}\mathcal{E}_t^H \xleftrightarrow{\sim} \text{mod-}C_0\mathcal{E}_t^H \xleftrightarrow{\sim} \text{mod-}H_*\mathcal{E}_t^H \cong \text{mod-}\mathcal{E}_a^H.$$

Proof This follows from Proposition 5.3.8. ■

Remark 4.3.11 With reference to Remark 9.3.6, we note that if $\text{mod-}\mathcal{E}_t^H$ was a monoidal category, then $H_*\mathcal{E}_t^H$ can be shown to have the same monoidal structure as \mathcal{E}_a^H . It would follow that the comparison between \mathcal{E}_t^H and \mathcal{E}_a^H would preserve the monoidal product on these categories. Hence, the zig-zag of the above result would be a zig-zag of strong monoidal equivalences by Proposition 5.3.10. We would then be able to conclude that the zig-zag between $S_H\text{-mod}$ and $dg\mathbb{Q}W_GH\text{-mod}$ would consist of symmetric monoidal equivalences.

Corollary 4.3.12 *If G is a finite group, then the model category of rational G -spectra is Quillen equivalent to the algebraic model for rational G -spectra:*

$$dg\mathcal{A}(G) = \prod_{(H) \leq G} dg\mathbb{Q}W_GH\text{-mod}.$$

Proof We begin with Theorem 4.2.4, which splits rational G -spectra into the product $\prod_{(H) \leq G} L_{E(H)}G\mathcal{M}_{\mathbb{Q}}$. Applying Proposition 4.2.12 to each factor of this category allows us to move to $\prod_{(H) \leq G} S_H\text{-mod}$. Next we use Theorem 4.3.2 to move to modules over a spectral category, $\prod_{(H) \leq G} \text{mod-}\mathcal{E}_{top}^H$. We move to algebra $(\prod_{(H) \leq G} \text{mod-}\mathcal{E}_t^H)$ with Theorem 4.3.3, and then use Corollary 4.3.10 to get to the category $\prod_{(H) \leq G} \text{mod-}\mathcal{E}_a^H$. Finally we use Proposition 4.1.8 to complete the result. ■

Remark 4.3.13 We can relate our work to [GM95, Appendix A] (specifically A.15 and Theorem A.16) as follows:

$$[X, Y]_{\mathbb{Q}}^{G|H} \cong [X, \widehat{f}_HY]_{\mathbb{Q}}^G \cong [X, e_HY]_{\mathbb{Q}}^G$$

where the first entry is maps in $\text{Ho } L_{E(H)}G\mathcal{M}_{\mathbb{Q}}$ and \widehat{f}_H is fibrant replacement in this model category. The G -spectra \widehat{f}_HY and e_HY are rationally equivalent which gives

us the second isomorphism in the above. Hence we can use the results of that paper to write the following, where $[i^*\Phi^H X, i^*\Phi^H Y]_{\mathbb{Q}}$ is the collection of rational homotopy classes of non-equivariant maps of naive $W_G H$ -spectra, which is a $\mathbb{Q}W_G H$ -module.

$$[X, Y]_{\mathbb{Q}}^G \cong \bigoplus_{(H)} [X, e_H Y]_{\mathbb{Q}}^G \cong \bigoplus_{(H)} \{[i^*\Phi^H X, i^*\Phi^H Y]\}^{W_G H}$$

Chapter 5

Enriched categories

The methods of [GS] rely heavily on highly structured model categories. We will need to use enrichments, tensorings, cotensorings and algebra structures, so we go through the definitions and basic constructions here. This is largely a service chapter where we introduce the language needed for the comparisons of Section 4.3 and Chapter 9. The first three sections are mainly providing definitions and results from [Kel05], [Hov99, Chapter 4] and [SS03b] respectively. The final section is the Morita equivalence: that one can replace a category by modules over an endomorphism ringoid. In [GS] they state that one can do so in a monoidal fashion, we have given full details of this result in Theorems 5.4.3 and 5.3.9. Thus our work in this chapter is mostly in relating the various definitions and giving more details on monoidal considerations. As well as the above-named references we will also make use of [Bor94], from which we take some more technical results on enriched categories. For the model category considerations we use [DS07] and [Dug06], which overlap somewhat.

5.1 ν -Categories

Throughout ν will be a monoidal category (symmetric when necessary), the unit of ν will be \mathcal{I} and the product will be \otimes . When needed, we will let ν have an internal function object $\underline{\text{Hom}}$, so that ν will be a closed monoidal category (see [Kel05, Section 1.5]).

Definition 5.1.1 A ν -category \mathcal{A} is a class of objects with a **Hom-object** $\mathcal{A}(A, B)$ in ν for each pair of objects. For each triple of objects there is a **composition law**: $M: \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, C)$, a morphism in ν . For each object there is an **identity element**: $j: \mathcal{I} \rightarrow \mathcal{A}(A, A)$, a morphism in ν . These must satisfy the usual five-fold associativity diagram and a pair of triangles describing the identity elements.

Definition 5.1.2 A ν -functor $T: \mathcal{A} \rightarrow \mathcal{B}$ of ν -categories is a functor $T: \text{Ob } \mathcal{A} \rightarrow \text{Ob } \mathcal{B}$ with a map in ν : $T = T_{A,B}: \mathcal{A}(A, B) \rightarrow \mathcal{B}(TA, TB)$ for each pair of objects of \mathcal{A} . These maps must satisfy the relations $TM = M(T \otimes T)$ and $Tj = j$.

When ν is the category of abelian groups, a ν -category is also known as a **ring with many objects**. A ν -category \mathcal{A} with one object a , is just a ring: composition gives multiplication on the abelian group $\mathcal{A}(a, a)$. Hence we will also refer to enriched categories as **ringoids** and a ν -functor is then a map of ringoids.

Definition 5.1.3 For ν -functors $T, S: \mathcal{A} \rightarrow \mathcal{B}$, a ν -**natural transformation** $\alpha: T \rightarrow S$ is an $\text{Ob } \mathcal{A}$ -indexed family of **components** $\alpha_A: \mathcal{T} \rightarrow \mathcal{B}(TA, SA)$ satisfying the naturality condition (in ν) $M(\alpha_B \otimes T)l^{-1} = M(S \otimes \alpha_A)r^{-1}$ where l and r are the left and right unit isomorphisms of ν .

The composite of β and α has components $M(\beta_A \otimes \alpha_A)$. The composite $Q\alpha$ has components $Q\alpha_A$, and αP has components $(\alpha P)_D = \alpha_{PD}$.

Definition 5.1.4 A ν -**adjunction** of ν -categories is an adjunction (F, G) consisting of ν -functors together with an isomorphism in ν : $\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB)$.

The above material was taken from [Kel05, Section 1.2].

Definition 5.1.5 If ν is a symmetric monoidal category and \mathcal{A} is a ν -category then $\mathcal{A} \times \mathcal{A}$ is a ν -category with

$$\mathcal{A} \times \mathcal{A}((a, b), (c, d)) := \mathcal{A}(a, c) \otimes \mathcal{A}(b, d)$$

and composition defined using the symmetry of ν as follows

$$\begin{array}{c} (\mathcal{A}(b, c) \otimes \mathcal{A}(y, z)) \otimes (\mathcal{A}(a, b) \otimes \mathcal{A}(x, y)) \\ \downarrow \text{Id} \otimes T \otimes \text{Id} \\ (\mathcal{A}(b, c) \otimes \mathcal{A}(a, b)) \otimes (\mathcal{A}(y, z) \otimes \mathcal{A}(x, y)) \\ \downarrow \\ \mathcal{A}(a, c) \otimes \mathcal{A}(x, z). \end{array}$$

The following is taken from [Day70, Page 2].

Definition 5.1.6 Let ν be a symmetric monoidal category. A **monoidal ν -category** is a ν -category \mathcal{A} with a ν -functor (the monoidal product) $\wedge: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, with a unit object $\mathbb{S} \in \text{Ob } \mathcal{A}$ and ν -natural isomorphisms for associativity and the unit. If there is a ν -natural isomorphism between the ν -functors \wedge and $\wedge \circ T$ where T interchanges the factors of $\mathcal{A} \times \mathcal{A}$ then \mathcal{A} is called **symmetric**. A **monoidal ν -functor** is a ν -functor that preserves the monoidal structure. That is, $\Psi: \mathcal{A} \rightarrow \mathcal{B}$, a ν -functor, is **monoidal** if $\wedge_{\mathcal{B}} \circ (\Psi \times \Psi) \cong \Psi \circ \wedge_{\mathcal{A}}$ as ν -functors. If \mathcal{A} and \mathcal{B} are symmetric then we can also require that Ψ respects the symmetry isomorphisms.

We spell out part of this definition, for any a, b, c and d in \mathcal{A} we have a map

$$(\mathcal{A} \times \mathcal{A})((a, b), (c, d)) := \mathcal{A}(a, c) \otimes \mathcal{A}(b, d) \longrightarrow \mathcal{A}(a \wedge b, c \wedge d)$$

that satisfies various unital and associativity diagrams. The symmetry of ν is used to relate the composition in $\mathcal{A} \times \mathcal{A}$ to the composition in \mathcal{A} . We can now use the construction of [Day70], as taken from [GS, Definition 3.5 and A.2].

Definition 5.1.7 Assume that \mathcal{A} is a symmetric monoidal ν -category. Then we can define a **box product** of ν -functors $T, S: \mathcal{A} \rightarrow \nu$.

$$T \square_{\mathcal{A}} S(a) = \int^{x,y} \mathcal{A}(x \wedge y, a) \otimes T(x) \otimes S(y)$$

The relevant coequaliser defining the above coend uses the following maps, first we have the ‘action’ on T and S .

$$\mathcal{A}(u \wedge v, a) \otimes \mathcal{A}(x, u) \otimes \mathcal{A}(y, v) \otimes T(x) \otimes S(y) \rightarrow \mathcal{A}(u \wedge v, a) \otimes T(u) \otimes S(v).$$

The second uses the monoidal structure on \mathcal{A} : $\mathcal{A}(x, u) \otimes \mathcal{A}(y, v) \rightarrow \mathcal{A}(x \wedge y, u \wedge v)$, followed by composition. There is an external product on the category of ν -functors $\mathcal{A} \rightarrow \nu$. Take two such functors T and S , then $T \bar{\wedge} S: \mathcal{A} \times \mathcal{A} \rightarrow \nu$ is a ν -functor with $(T \bar{\wedge} S)(a, b) = T(a) \otimes S(b)$. We can describe the box product as the left Kan extension of \otimes along \wedge , the monoidal product of \mathcal{A} .

Proposition 5.1.8 Let $F: \nu \rightarrow \mu$ be a symmetric monoidal functor and \mathcal{C} be a symmetric monoidal ν -category. Then FC is a symmetric monoidal μ -category.

Proof We define FC to have the same class of objects as \mathcal{C} . On maps we define $(FC)(a, b) = F(\mathcal{C}(a, b))$. The monoidality of F gives us the composition rule and identity elements. The monoidal structure is created from the monoidal structure on \mathcal{C} , but to prove that the monoidal product is a μ -functor we must use the symmetry of F . The required unital and associativity μ -natural isomorphisms of the monoidal product then follow from those in \mathcal{C} . This result is an application of [DS07, Proposition A.3(b)] with monoidal structures considered. ■

The need for symmetry in the above will cause us difficulty in Chapter 9 when we create an enriched category $D(\phi^* N \tilde{\mathbb{Q}} \mathcal{E}_{top})$ by applying the functor D to the enriched category $\phi^* N \tilde{\mathbb{Q}} \mathcal{E}_{top}$.

We now give a few results which give us some rules we can use when working with coends or ends. Note that the Yoneda lemma states that the end $\int_c \underline{\text{Hom}}(\mathcal{C}(x, c), F(c))$ exists, whereas Lemma 5.1.10 and Corollary 5.1.11 assume that the relevant coend exists. Of course, we will be working in the context of model categories, whence the ends and coends we consider below will always exist.

Lemma 5.1.9 (Enriched Yoneda) Take ν to be a symmetric closed monoidal category and let \mathcal{C} be a small ν -category. Then for any $x \in \mathcal{C}$ and any ν -functor $F: \mathcal{C} \rightarrow \nu$ the ν -object $\nu\text{-Nat}(\mathcal{C}(x, -), F) := \int_c \underline{\text{Hom}}(\mathcal{C}(x, c), F(c))$ exists and is naturally isomorphic to $F(x)$.

Proof This lemma is [Bor94, Proposition 6.3.5]. ■

Lemma 5.1.10 *Consider a ν -functor $F: \mathcal{C} \rightarrow \nu$ between ν -categories (with \mathcal{C} small) and let $x \in \mathcal{C}$, then whenever the following coend exists, there is an isomorphism (natural in F and x) in ν :*

$$\int^{c \in \mathcal{C}} F(c) \otimes \mathcal{C}(c, x) \xrightarrow{\cong} F(x).$$

Proof We write the coend as a coequaliser of coproducts

$$\coprod_{c, d} F(c) \otimes \mathcal{C}(c, d) \otimes \mathcal{C}(d, x) \rightrightarrows \coprod_e F(e) \otimes \mathcal{C}(e, x).$$

One arrow corresponds to composition and the other to the action of $\mathcal{C}(c, d)$ on $F(c)$

$$F(c) \otimes \mathcal{C}(c, d) \rightarrow F(c) \otimes \nu(F(c), F(d)) \rightarrow F(d).$$

The action map then induces a map in ν : $\coprod_c F(c) \otimes \mathcal{C}(c, x) \rightarrow F(x)$, that this coequalises the two maps follows from the definition of a ν -functor. Given any test object T for this coequaliser we can use factor x of the given map $\alpha: \coprod_c F(c) \otimes \mathcal{C}(c, x) \rightarrow T$ and the composite $F(x) \cong F(x) \otimes \mathcal{I} \rightarrow F(x) \otimes \mathcal{C}(x, x)$ to give a map $F(x) \rightarrow T$. It follows easily that this map is unique and satisfies the required commutativity condition. Hence $F(x)$ is isomorphic to the coend. ■

Corollary 5.1.11 *Consider a ν -functor $F: \mathcal{C}^{op} \rightarrow \nu$ between ν -categories (with \mathcal{C} small) and let $x \in \mathcal{C}$, then whenever the following coend exists, there is an isomorphism (natural in F and x) in ν :*

$$\int^{c \in \mathcal{C}} F(c) \otimes \mathcal{C}(x, c) \xrightarrow{\cong} F(x).$$

5.2 \mathcal{C} -Modules

We now wish to move from enrichments to tensorings of categories (and then to tensorings, cotensorings and enrichments all at the same time). To avoid confusion with the previous information and to match the notation of [Hov99, Chapter 4] we let \mathcal{C} be a symmetric monoidal category ([Hov99, Definition 4.1.4]) with product \otimes and unit $1_{\mathcal{C}}$.

Definition 5.2.1 A **\mathcal{C} -module** is a category \mathcal{D} with a functor $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$. The symmetry of \mathcal{C} allows us to use left-modules and right-modules interchangeably. A **\mathcal{C} -module functor** between \mathcal{C} -modules \mathcal{D} and \mathcal{E} is a functor $F: \mathcal{D} \rightarrow \mathcal{E}$ with a natural isomorphism $c \otimes Fd \cong F(c \otimes d)$ satisfying the following pair of coherence conditions. Let c and c' be objects of \mathcal{C} and d an object of \mathcal{D} . The first condition is that the two ways of getting from $L(c \otimes (c' \otimes d))$ to $c \otimes (c' \otimes Ld)$ must be equal. The second is that the two ways to get from $L(d \otimes 1_{\mathcal{C}})$ to Ld must agree.

A **(symmetric) \mathcal{C} -algebra** is a category \mathcal{D} with a (symmetric) monoidal structure and a strong (symmetric) monoidal functor $i_{\mathcal{D}}: \mathcal{C} \rightarrow \mathcal{D}$. A **(symmetric) \mathcal{C} -algebra functor** between \mathcal{C} -algebras \mathcal{D} and \mathcal{E} is a strong (symmetric) monoidal functor $F: \mathcal{D} \rightarrow \mathcal{E}$ with a monoidal natural isomorphism $F \circ i_{\mathcal{D}} \rightarrow i_{\mathcal{E}}$.

This definition is an abbreviation of [Hov99, Definitions 4.1.6 – 4.1.9]. We now want to add still more structure: to consider categories enriched, tensored and cotensored over another category. We let \mathcal{C} be a closed (symmetric) monoidal category [Hov99, 4.1.13] with product \otimes and unit $1_{\mathcal{C}}$, the internal function object of \mathcal{C} will be $F_{\mathcal{C}}$ (recall that the adjective closed means that we have an adjunction of two variables rather than just a bifunctor).

Definition 5.2.2 A closed \mathcal{C} -module is a category \mathcal{D} with an adjunction of two variables $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$. Thus we have three bifunctors

$$\begin{aligned} \otimes : \quad \mathcal{C} \times \mathcal{D} &\longrightarrow \mathcal{D} \\ \underline{\text{Hom}} : \quad \mathcal{C}^{op} \times \mathcal{D} &\longrightarrow \mathcal{D} \\ \underline{\text{Hom}}_{\mathcal{C}} : \quad \mathcal{D}^{op} \times \mathcal{D} &\longrightarrow \mathcal{C}. \end{aligned}$$

In [Hov99] the notation Hom_r is used for $\underline{\text{Hom}}_{\mathcal{C}}$ and Hom_l for $\underline{\text{Hom}}$. We also have the adjunctions $\mathcal{C}(c, \underline{\text{Hom}}_{\mathcal{C}}(d, d')) \cong \mathcal{D}(c \otimes d, d') \cong \mathcal{D}(d, \underline{\text{Hom}}(c, d'))$. We call \otimes the **tensor** operation, $\underline{\text{Hom}}$ the **cotensor** and $\underline{\text{Hom}}_{\mathcal{C}}$ the **enrichment**.

Thus a closed \mathcal{C} -module is a \mathcal{C} -category, with the enrichment given by $\underline{\text{Hom}}_{\mathcal{C}}$.

Lemma 5.2.3 Let \mathcal{D} be a closed \mathcal{C} -module, then for any d, d' in \mathcal{D} and c in \mathcal{C} , there are canonical isomorphisms in \mathcal{C}

$$\mathcal{C}(c, \underline{\text{Hom}}_{\mathcal{C}}(d, d')) \cong \underline{\text{Hom}}_{\mathcal{C}}(d \otimes c, d') \cong \underline{\text{Hom}}_{\mathcal{C}}(d, \underline{\text{Hom}}(c, d'))$$

which, after applying $\mathcal{C}(1_{\mathcal{C}}, -)$, reduce to the isomorphisms of the definition above.

Proof This is [Dug06, Lemma A.2]. ■

Lemma 5.2.4 Take an adjunction $L : \mathcal{D} \rightleftarrows \mathcal{E} : R$ between closed \mathcal{C} -modules, then the following statements are equivalent:

- (i). There are natural isomorphisms in \mathcal{E} , $c \otimes Ld \cong L(c \otimes d)$, which reduce to the canonical isomorphisms for $c = 1_{\mathcal{C}}$.
- (ii). There are natural isomorphisms in \mathcal{D} , $\underline{\text{Hom}}(c, Re) \cong R \underline{\text{Hom}}(c, e)$ which reduce to the canonical isomorphisms for $c = 1_{\mathcal{C}}$.
- (iii). There are natural isomorphisms in \mathcal{C} , $\underline{\text{Hom}}_{\mathcal{C}}(d, Re) \cong \underline{\text{Hom}}_{\mathcal{C}}(Ld, e)$ which after applying $\mathcal{C}(1_{\mathcal{C}}, -)$ reduce to the adjunction between L and R .

Proof This is [Dug06, Lemma A.7] and we give a proof of this simple result. Take any c, d, e in $\mathcal{C}, \mathcal{D}, \mathcal{E}$ respectively. Then

$$\begin{aligned} \mathcal{E}(c \otimes Ld, e) &\cong \mathcal{E}(Ld, \underline{\text{Hom}}(c, e)) \cong \mathcal{D}(d, R \underline{\text{Hom}}(c, e)) \\ \mathcal{E}(L(c \otimes d), e) &\cong \mathcal{D}(c \otimes d, Re) \cong \mathcal{D}(d, \underline{\text{Hom}}(c, Re)) \end{aligned}$$

so since an object in a category is determined (up to natural isomorphism) by the maps out of (or the maps into) that object, we can see that the first two conditions are

equivalent. The next two collections of isomorphisms show (respectively) that the first and last conditions are equivalent and that the second and third are equivalent.

$$\begin{aligned} \mathcal{C}(c, \underline{\text{Hom}}_{\mathcal{C}}(d, Re)) &\cong \mathcal{D}(c \otimes d, Re) \cong \mathcal{E}(L(c \otimes d), e) \\ \mathcal{C}(c, \underline{\text{Hom}}_{\mathcal{C}}(Ld, e)) &\cong \mathcal{E}(c \otimes Ld, e) \\ \\ \mathcal{C}(c, \underline{\text{Hom}}_{\mathcal{C}}(d, Re)) &\cong \mathcal{D}(c \otimes d, Re) \cong \mathcal{D}(d, \underline{\text{Hom}}(c, Re)) \\ \mathcal{C}(c, \underline{\text{Hom}}_{\mathcal{C}}(Ld, e)) &\cong \mathcal{E}(Ld, \underline{\text{Hom}}(c, e)) \cong \mathcal{D}(d, R \underline{\text{Hom}}(c, e)) \end{aligned}$$

The statement about reductions are similarly routine. ■

Definition 5.2.5 An **adjunction of closed \mathcal{C} -modules** is an adjoint pair such that the left adjoint is a \mathcal{C} -module functor. A **closed (symmetric) \mathcal{C} -algebra** is a category \mathcal{D} with a closed (symmetric) monoidal structure and a strong (symmetric) monoidal adjunction $i_{\mathcal{D}} : \mathcal{C} \rightleftarrows \mathcal{D} : j_{\mathcal{D}}$ [Hov99, Definition 4.1.14]. An **adjunction of closed (symmetric) \mathcal{C} -algebras** between \mathcal{C} -algebras \mathcal{D} and \mathcal{E} is a strong (symmetric) monoidal adjunction $L : \mathcal{D} \rightleftarrows \mathcal{E} : R$ with a monoidal natural isomorphism $L \circ i_{\mathcal{D}} \rightarrow i_{\mathcal{E}}$ (so L is a \mathcal{C} -algebra functor).

Lemma 5.2.6 *Take a strong monoidal adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$. Then a \mathcal{D} -module category M can be given the structure of a \mathcal{C} -module category by setting*

$$X \otimes c = X \otimes Lc, \quad F_{\mathcal{C}}(c, Y) = F(Lc, Y) \quad \text{and} \quad \underline{\text{Hom}}_{\mathcal{C}}(X, Y) = R \underline{\text{Hom}}_{\mathcal{D}}(X, Y)$$

note that the direction of the adjunction is essential to this lemma.

Proof This is [Dug06, Lemma A.5]. ■

Corollary 5.2.7 *If there is a strong (symmetric) monoidal adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ between closed (symmetric) monoidal categories then \mathcal{C} and \mathcal{D} are closed (symmetric) \mathcal{C} -algebras and (L, R) is an adjunction of closed (symmetric) \mathcal{C} -algebras.*

We now link these structures back to the definitions of [Kel05].

Proposition 5.2.8 *If $L : \mathcal{D} \rightleftarrows \mathcal{E} : R$ is an adjunction of closed \mathcal{C} -modules then both L and R are \mathcal{C} -functors.*

Proof Showing this for the right adjoint takes a rather large diagram chase, see [Dug06, Proposition A.9] for a proof of the statement. That the left adjoint is a \mathcal{C} -functor follows from the statement for the right adjoint and [Bor94, Proposition 6.7.2]. For $d \in \mathcal{D}$ there is a unit map $d \rightarrow RLd$ in \mathcal{D} and for $e \in \mathcal{E}$ there is a counit map $LRe \rightarrow e$ in \mathcal{E} . The \mathcal{C} -structures on L and R arise from these maps as below.

$$\begin{aligned} \underline{\text{Hom}}_{\mathcal{C}}(d', d) &\rightarrow \underline{\text{Hom}}_{\mathcal{C}}(d', RLd) \cong \underline{\text{Hom}}_{\mathcal{C}}(Ld', Ld) \\ \underline{\text{Hom}}_{\mathcal{C}}(e, e') &\rightarrow \underline{\text{Hom}}_{\mathcal{C}}(LRe, e') \cong \underline{\text{Hom}}_{\mathcal{C}}(Re, Re') \end{aligned}$$
■

Remark 5.2.9 Let \mathcal{C} be a (symmetric) monoidal category, then we can give \mathcal{C} the structure of a closed \mathcal{C} -module or a closed (symmetric) \mathcal{C} -algebra. If \mathcal{D} is a closed (symmetric) \mathcal{C} -algebra, then it is certainly a closed \mathcal{C} -module. So now we can ask: when is a closed \mathcal{C} -module \mathcal{D} a closed (symmetric) \mathcal{C} -algebra? Obviously the primary requirement is that \mathcal{D} should be a (symmetric) monoidal category, but we also need an associativity relation: $c \otimes (d \wedge d') \cong (c \otimes d) \wedge d'$ satisfying the appropriate coherence diagrams. Then we have a strong monoidal (symmetric) adjunction

$$(-) \otimes 1_{\mathcal{D}} : \mathcal{C} \rightleftarrows \mathcal{D} : \underline{\mathrm{Hom}}_{\mathcal{C}}(1_{\mathcal{D}}, -)$$

making \mathcal{D} into a closed (symmetric) \mathcal{C} -algebra. This is precisely analogous to the case of modules and algebras over a ring. When considering model structures the only additional condition is to require that the adjunction is a Quillen pair, which is automatic when $1_{\mathcal{D}}$ is cofibrant.

5.3 Modules over an Enriched Category

This section gives the language needed to compare our categories of G -spectra to algebra, that is, it allows us to make use of the results of [Shi07b]. We show how to move from spectra to algebra in Section 9.3. To help the notation we fix our ‘base category’ to be the category of symmetric spectra, thus we state all definitions and results in terms enrichments over symmetric spectra. It should be clear how to replace the category of symmetric spectra with any other closed symmetric monoidal model category (which satisfies the monoid axiom) in the following. In particular we will later use Sp_+^{Σ} , $Sp^{\Sigma}(\mathbf{sQ}\text{-mod})$, $Sp^{\Sigma}(dg\mathbf{Q}\text{-mod}_+)$ and $dg\mathbf{Q}\text{-mod}$ in the place of symmetric spectra. Note that to prove Theorem 5.3.9 (the main result of this section) we must either assume the unit is cofibrant or assume that smashing with a cofibrant object preserves weak equivalences.

We now give [SS03b, Definition 3.5.1], which takes the notion of a closed Sp^{Σ} -module and adds model structure conditions.

Definition 5.3.1 A **spectral model category** is a model category \mathcal{C} which is a closed Sp^{Σ} -module and the action map $\otimes : Sp^{\Sigma} \times \mathcal{C} \rightarrow \mathcal{C}$ is a Quillen bifunctor such that $\widehat{\mathbb{S}} \otimes X \rightarrow \mathbb{S} \otimes X$ is a weak equivalence for all cofibrant $X \in M$. A **spectral left Quillen functor** is a left Quillen functor that is also a Sp^{Σ} -module functor. A **spectral right Quillen functor** is a right Quillen functor $R : \mathcal{C} \rightarrow \mathcal{D}$ between spectral model categories that preserves (up to natural isomorphism in \mathcal{D}) the cotensor operation: $\underline{\mathrm{Hom}}(K, RX) \cong R \underline{\mathrm{Hom}}(K, X)$ and this isomorphism must be coherent in the sense that the two maps from $\underline{\mathrm{Hom}}(L, \underline{\mathrm{Hom}}(K, RX))$ to $R \underline{\mathrm{Hom}}(L, \underline{\mathrm{Hom}}(K, X))$ must be equal.

A spectral model category is a Sp^{Σ} -model category in the sense of [Hov99, Definition 4.2.18]. As (briefly) mentioned after [Hov99, Definition 4.1.12] the correct notion of a ‘map of spectral model categories’ is a spectral adjunction of categories as defined in [SS03b, Definition 3.9.2] which we give below.

Definition 5.3.2 Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjoint pair between spectral model categories \mathcal{C} and \mathcal{D} . A **spectral adjunction** is an adjunction of closed Sp^Σ -modules. We call such a pair a **spectral Quillen pair** if the functors (L, R) are also a Quillen pair. A **spectral Quillen equivalence** is a spectral Quillen pair that is also a Quillen equivalence.

A **closed (symmetric) monoidal spectral model category** is a closed (symmetric) Sp^Σ -algebra such that the adjunction $Sp^\Sigma \rightleftarrows \mathcal{D}$ is a Quillen pair. An **adjunction of closed (symmetric) monoidal spectral model categories** is an adjunction of closed (symmetric) Sp^Σ -algebras that is a Quillen pair.

Thus, a spectral adjunction is an adjunction with a natural isomorphism of symmetric spectra $\underline{\text{Hom}}_{\mathcal{C}}(A, RX) \cong \underline{\text{Hom}}_{\mathcal{D}}(LA, X)$. Furthermore, L is a spectral left Quillen functor and R is spectral right Quillen functor. We introduce some language for categories enriched over symmetric spectra which don't necessarily have model structures or aren't tensored or cotensored over symmetric spectra.

Definition 5.3.3 A **spectral category** ([SS03b, Definition 3.3.1]) is a category \mathcal{O} enriched over the category of symmetric spectra Sp^Σ , i.e. \mathcal{O} is an Sp^Σ -category. A **spectral functor** is an Sp^Σ -functor and a **spectral adjunction** is an Sp^Σ -adjunction. A (right) **\mathcal{O} -module** is a contravariant spectral functor $\mathcal{O} \rightarrow Sp^\Sigma$, these modules form a category denoted $\text{mod-}\mathcal{O}$. A **morphism of \mathcal{O} -modules** is an Sp^Σ -natural transformation.

We spell out the above requirements for an \mathcal{O} -module M . For every object of \mathcal{O} we have a symmetric spectrum $M(o)$ and we must have coherently unital and associative maps of symmetric spectra $M(o) \wedge \mathcal{O}(o', o) \rightarrow M(o')$ for pairs of objects o, o' in \mathcal{O} . A morphism of such modules is a collection of maps of symmetric spectra $M(o) \rightarrow N(o)$ which are strictly compatible with the above action.

Definition 5.3.4 For each $o \in \mathcal{O}$ we have a **free module** ([SS03b, Definition 3.3.1]), F_o . This is an \mathcal{O} -module defined by $F_o(o') = \mathcal{O}(o', o)$ with \mathcal{O} -action given by composition.

Remark 5.3.5 The definitions above fit neatly into the framework of [Bor94, Section 6.2], If \mathcal{O} is an Sp^Σ -category then so is \mathcal{O}^{op} , [Bor94, Proposition 6.2.2]. Then we have the following definitions from [Bor94, Proposition 6.3.1]:

$$\begin{aligned} \text{Category of left } \mathcal{O}\text{-modules, } \mathcal{O}\text{-mod} &= Sp^\Sigma[\mathcal{O}, Sp^\Sigma] \\ \text{Category of right } \mathcal{O}\text{-modules, } \text{mod-}\mathcal{O} &= Sp^\Sigma[\mathcal{O}^{op}, Sp^\Sigma] \end{aligned}$$

the categories of covariant and contravariant spectral functors from \mathcal{O} to Sp^Σ . Morphisms in these categories are the Sp^Σ -natural transformations.

Theorem 5.3.6 Let \mathcal{O} be a spectral category, then the category of \mathcal{O} -modules with object-wise weak equivalences of Sp^Σ , object-wise fibrations of Sp^Σ and cofibrations as necessary gives a cofibrantly generated spectral model structure. The collection of free modules give a set of compact generators.

Proof This is part of [SS03b, Theorem A.1.1]. ■

We take the definition below from [SS03b, Section A.1].

Definition 5.3.7 For a spectral functor of spectral categories $\Psi: \mathcal{O} \longrightarrow \mathcal{R}$, we have a **restriction of scalars** functor

$$\Psi^*: \text{mod-}\mathcal{R} \longrightarrow \text{mod-}\mathcal{O}, \quad M \mapsto M \circ \Psi.$$

This has a left adjoint $(-) \wedge_{\mathcal{O}} \mathcal{R}$, the **extension of scalars** functor. It is defined as an enriched coend and is similar to many other constructions in this work,

$$N \wedge_{\mathcal{O}} \mathcal{R} = \int^{\mathcal{O}} N(o) \wedge F_{\Psi(o)}$$

we give a construction below as a coequaliser of a pair of \mathcal{R} -module homomorphisms.

$$\bigvee_{o,p \in \mathcal{O}} N(p) \wedge \mathcal{O}(o,p) \wedge F_{\Psi(o)} \rightrightarrows \bigvee_{q \in \mathcal{O}} N(q) \wedge F_{\Psi(q)}$$

We call Ψ a **stable equivalence** if each $\Psi_{o,p}: \mathcal{O}(o,p) \rightarrow \mathcal{R}(o,p)$ is a weak equivalence of symmetric spectra.

Proposition 5.3.8 *For a spectral functor of spectral categories $\Psi: \mathcal{O} \longrightarrow \mathcal{R}$ there is a Quillen pair*

$$(-) \wedge_{\mathcal{O}} \mathcal{R} : \text{mod-}\mathcal{O} \rightleftarrows \text{mod-}\mathcal{R} : \Psi^*.$$

Furthermore if Ψ is a stable equivalence this adjoint pair is a Quillen equivalence.

Proof This is part of [SS03b, Theorem A.1.1]. ■

We now add an extra condition on \mathcal{O} to ensure that $\text{mod-}\mathcal{O}$ is in fact a closed monoidal model category. Note that we will use the fact that in Sp^{Σ} (or Sp_+^{Σ}) smashing with a cofibrant object preserves weak equivalences.

Theorem 5.3.9 *If the category \mathcal{O} is a symmetric monoidal Sp^{Σ} -category then the category $\text{mod-}\mathcal{O}$ is a closed symmetric monoidal model category satisfying the monoid axiom.*

Proof This is [GS, Proposition 3.7], adjusted to the setting of an Sp^{Σ} -category, with full details given. The monoidal product is the box product

$$M \square N(o) = \int^{p,q} \mathcal{O}(o, p \wedge q,) \otimes M(p) \otimes N(q).$$

We let \wedge be the monoidal product of \mathcal{O} and \mathbb{S} the unit. The unit of $\text{mod-}\mathcal{O}$ is the module $\underline{\text{Hom}}_{Sp^{\Sigma}}(-, \mathbb{S})$, which is easy to prove using Corollary 5.1.11. It is then a matter of formality that the internal function object exists, it is defined as

$$\underline{\text{Hom}}_{\square}(M, N)(o) = Sp^{\Sigma}\text{-Nat}(M, N(o \wedge -)) = \int_p \underline{\text{Hom}}(M(p), N(o \wedge p)).$$

Now we show that the pushout product and monoid axiom hold, to do so we must understand the generating cofibrations. Thus we write out part of the proof of [SS03a, Theorem 6.1]. This proves the model structure exists by identifying $\text{mod-}\mathcal{O}$ with a category of algebras over a triple.

Let $\mathbb{I}_{\mathcal{O}}$ be the Sp^{Σ} -category with the same objects as \mathcal{O} and maps given by $\mathbb{I}_{\mathcal{O}}(o, o) = \mathbb{S}$ and a point otherwise. There is a canonical map of Sp^{Σ} -categories $u: \mathbb{I}_{\mathcal{O}} \rightarrow \mathcal{O}$, which is given by the unit map $\mathbb{I}_{\mathcal{O}}(o, o) \rightarrow \mathcal{O}(o, o)$. The adjoint pair of restriction and extension of scalars (Definition 5.3.7), gives the required triple T ([Mac71, Chapter VI]) on $\text{mod-}\mathbb{I}_{\mathcal{O}}$ and the algebras over this triple are \mathcal{O} -modules. Now $\text{mod-}\mathbb{I}_{\mathcal{O}}$ is simply an \mathcal{O} -indexed product of copies of Sp^{Σ} hence we can give it the product model structure [Hov99, 1.1.6]. The generating (acyclic) cofibrations for $\text{mod-}\mathbb{I}_{\mathcal{O}}$ are given by maps $f: A \rightarrow B$ concentrated in factor o where f is a generating (acyclic) cofibration for Sp^{Σ} . Hence a generating (acyclic) cofibration for $\text{mod-}\mathcal{O}$ has form $A \wedge F_o \rightarrow B \wedge F_o$ where f is a generating (acyclic) cofibration for Sp^{Σ} . In fact this argument shows that smashing with a free module takes (acyclic) cofibrations of Sp^{Σ} to (acyclic) cofibrations of $\text{mod-}\mathcal{O}$.

The monoid axiom follows from the monoid axiom for Sp^{Σ} (for the positive case see [MMSS01, Theorem 14.2]). Take a generating acyclic cofibration $A \wedge F_o \rightarrow B \wedge F_o$ and a module M , for $p \in \mathcal{O}$ we have $((A \wedge F_o) \square M)(p) = A \wedge (F_o \square M)(p)$ hence $(A \wedge F_o) \square M \rightarrow (B \wedge F_o) \square M$ is a weak equivalence in $\text{mod-}\mathcal{O}$. Since pushouts and transfinite compositions of \mathcal{O} -modules are constructed object-wise, the rest of the monoid axiom follows.

The pushout product axiom for $\text{mod-}\mathcal{O}$ uses the isomorphism $(A \wedge F_o) \square (B \wedge F_p) \cong (A \wedge B) \wedge F_{o \wedge p}$, and the pushout product axiom for symmetric spectra. We must also prove that for any cofibrant module M the map

$$\widehat{\mathcal{O}}(-, \mathbb{S}) \square M \rightarrow M$$

is a weak equivalence. We actually prove this result holds without the assumption that M is cofibrant. The unit of Sp^{Σ} is cofibrant, hence so is $\mathcal{O}(-, \mathbb{S})$ and there is nothing to check. In a category where the unit is not cofibrant, such as Sp^{Σ}_+ , an alternative proof exists whenever smashing with cofibrant objects preserves weak equivalence. The important step is in identifying $\widehat{\mathcal{O}}(-, \mathbb{S})$ with $\widehat{\mathbb{S}} \wedge \mathcal{O}(-, \mathbb{S})$, the latter is cofibrant as an \mathcal{O} -module since $\widehat{\mathbb{S}}$ is cofibrant in Sp^{Σ}_+ . The map

$$\widehat{\mathbb{S}} \wedge \mathcal{O}(-, \mathbb{S}) \rightarrow \mathcal{O}(-, \mathbb{S})$$

is a weak equivalence since $\widehat{\mathbb{S}} \wedge X \rightarrow X$ is a weak equivalence for any symmetric spectrum X . So we have reduced the problem to proving that $\widehat{\mathbb{S}} \wedge M \rightarrow M$ is a weak equivalence, but this is obvious from the above statements. \blacksquare

Proposition 5.3.10 *Let $\Psi: \mathcal{O} \rightarrow \mathcal{R}$ be a monoidal functor of monoidal spectral categories. Then $((-) \wedge_{\mathcal{O}} \mathcal{R}, \Psi^*)$ is a strong monoidal Quillen adjunction. If the categories and Ψ are also symmetric then the adjunction will be a strong symmetric monoidal adjunction.*

Proof That the left adjoint is strong monoidal is an easy exercise in manipulating coends. Thus the right adjoint has a weak monoidal structure and the left adjoint has the required model structure properties. When the unit is cofibrant this is all that is required. In the Sp_+^Σ case, the result follows from the identification of $\widehat{\mathcal{O}}(-, \mathbb{S})$ with $\widehat{\mathcal{S}} \wedge \mathcal{O}(-, \mathbb{S})$ and the isomorphism $\mathcal{O}(-, \mathbb{S}) \wedge_{\mathcal{O}} \mathcal{R} \cong \mathcal{R}(-, \mathbb{S})$. ■

Definition 5.3.11 Given a pair of spectral functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ between spectral categories there is a symmetric spectrum of spectral natural transformations. For brevity we call this the **spectrum of natural transformations** and it is defined as ([Bor94, Proposition 6.3.1])

$$Sp^\Sigma\text{-Nat}(F, G) = \int_c \underline{\text{Hom}}(Fc, Gc).$$

As an equaliser of products we have

$$Sp^\Sigma\text{-Nat}(F, G) \longrightarrow \prod_{c \in \mathcal{C}} \underline{\text{Hom}}(Fc, Gc) \rightrightarrows \prod_{c, c' \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{C}(c, c') \otimes Fc, Gc')$$

where $\underline{\text{Hom}}$ is the internal function object for symmetric spectra.

Remark 5.3.12 The above is the general definition, but when working with right modules over \mathcal{O} one must remember to account for the change in variance. So for right \mathcal{O} -modules M and N the **spectrum of morphisms** of \mathcal{O} -modules is given below.

$$Sp^\Sigma\text{-Nat}(M, N) \longrightarrow \prod_{c \in \mathcal{C}} \underline{\text{Hom}}(Mc, Nc) \rightrightarrows \prod_{c', c \in \mathcal{C}} \underline{\text{Hom}}(\mathcal{O}(c, c') \otimes Mc, Nc')$$

Hence $\text{mod-}\mathcal{O}$ is an Sp^Σ -category.

Proposition 5.3.13 *The category of \mathcal{O} -modules is tensored and cotensored over symmetric spectra.*

Proof The tensor and cotensor are given object-wise; so for a symmetric spectrum A and a (right) \mathcal{O} -module M

$$(A \otimes M)(o) = A \wedge M(o) \quad \underline{\text{Hom}}(A, M)(o) = \underline{\text{Hom}}(A, M(o)).$$

Now we show that these are compatible with the enrichment, so that:

$$Sp^\Sigma\text{-Nat}(M, \underline{\text{Hom}}(A, N)) \cong Sp^\Sigma\text{-Nat}(A \otimes M, N) \cong \underline{\text{Hom}}(A, Sp^\Sigma\text{-Nat}(M, N)).$$

Writing this out in terms of ends we require a pair of relations

$$\begin{aligned} \int_o \underline{\text{Hom}}(A \wedge M(o), N(o)) &\cong \int_o \underline{\text{Hom}}(M(o), \underline{\text{Hom}}(A, N(o))) \\ \int_o \underline{\text{Hom}}(A, \underline{\text{Hom}}(M(o), N(o))) &\cong \underline{\text{Hom}}(A, \int_o \underline{\text{Hom}}(M(o), N(o))) \end{aligned}$$

the first of which is certainly true and the second follows from [Bor94, Proposition 6.7.3]: the associativity of Sp^Σ ensures that $(A \wedge (-), \underline{\text{Hom}}(A, -))$ is a Sp^Σ -adjunction. ■

In truth the above enrichment, tensoring and cotensoring over symmetric spectra actually arises from giving $\text{mod-}\mathcal{O}$ the structure of a Sp^Σ -algebra. The result as stated below holds in greater generality, one can replace Sp^Σ by any other symmetric monoidal model category provided that either the unit is cofibrant or that smashing with a cofibrant object preserves weak equivalences.

Theorem 5.3.14 *If \mathcal{O} is a symmetric monoidal Sp^Σ -category, then there is a strong symmetric monoidal Quillen pair*

$$(-) \otimes F_{\mathbb{S}} : Sp^\Sigma \rightleftarrows \text{mod-}\mathcal{O} : \underline{\text{Hom}}(F_{\mathbb{S}}, -)$$

where $F_{\mathbb{S}}$ is the free module on the unit of \mathcal{O} . Thus $\text{mod-}\mathcal{O}$ is an Sp^Σ -algebra. This structure is compatible with the enrichment, tensoring and cotensoring above.

Proof We clearly have a symmetric strong monoidal pair and we have already shown that $(-) \otimes F_{\mathbb{S}}$ preserves (acyclic) cofibrations in Theorem 5.3.9. To recover the tensoring one simply needs to use the basic properties of coends (such as commutation with left adjoints and interchange of factors) and Corollary 5.1.11 to see $A \otimes M \cong (A \otimes F_{\mathbb{S}}) \square M$. For the cotensoring, the same kind of argument suffices to show $\underline{\text{Hom}}(A, M) \cong \underline{\text{Hom}}_{\square}(A \otimes F_{\mathbb{S}}, M)$, but we need the end version of Corollary 5.1.11 since we are now in a ‘right-handed’ case. Finally, the enrichment can be identified as $\underline{\text{Hom}}(M, N) \cong \underline{\text{Hom}}(F_{\mathbb{S}}, \underline{\text{Hom}}_{\square}(M, N))$. ■

Note that by the enriched Yoneda lemma (Lemma 5.1.9), $\underline{\text{Hom}}(F_{\mathbb{S}}, M) \cong M(\mathbb{S})$ for any \mathcal{O} -module M .

Remark 5.3.15 It is instructive to look at this result in a little more generality. For each $o \in \mathcal{O}$ there is a Quillen pair $(-) \wedge F_o : Sp^\Sigma \rightleftarrows \text{mod-}\mathcal{O} : \text{Ev}_o$ where $\text{Ev}_o(M) = M(o)$, the evaluation functor. To show that this is an adjoint pair we use the enriched Yoneda lemma (Lemma 5.1.9),

$$\text{mod-}\mathcal{O}(A \wedge F_o, M) \cong Sp^\Sigma(A, \underline{\text{Hom}}(F_o, M)) \cong Sp^\Sigma(A, M(o))$$

we can now use Theorem 5.3.9 to see that this is a Quillen pair. Alternatively, we can show this more directly: since fibrations and weak equivalences are defined object-wise, each Ev_o is a right Quillen functor.

5.4 Morita Equivalences

The theorem below is vital to our work, it allows us to replace G -spectra by modules over an Sp^Σ -category $\mathcal{E}(\mathcal{G})$.

Definition 5.4.1 Consider a set of objects \mathcal{G} in a spectral model category \mathcal{D} . We define $\mathcal{E}(\mathcal{G})$, the **endomorphism ringoid of \mathcal{G}** , to be the spectral category with object set \mathcal{G} and $\mathcal{E}(\mathcal{G})(g, g') = \underline{\text{Hom}}_{Sp^\Sigma}(g, g')$.

Definition 5.4.2 For a spectral model category \mathcal{D} and a full subcategory $\mathcal{E}(\mathcal{G})$ defined by an object set \mathcal{G} , there is a functor (sometimes called the tautological functor)

$$\underline{\text{Hom}}(\mathcal{G}, -) : \mathcal{D} \longrightarrow \text{mod-}\mathcal{E}(\mathcal{G})$$

defined by $\underline{\text{Hom}}(\mathcal{G}, d)(g) = \underline{\text{Hom}}_{Sp^\Sigma}(g, d)$ (this is [SS03b, Definition 3.9.1]). This has a left adjoint $- \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ and for a module M this is given by a coend $\int^{g \in \mathcal{G}} M(g) \wedge g$. This can also be written in terms of a coequaliser of coproducts:

$$\bigvee_{g, h \in \mathcal{G}} M(h) \wedge \mathcal{E}(\mathcal{G})(g, h) \wedge g \rightrightarrows \bigvee_{g \in \mathcal{G}} M(g) \wedge g.$$

Theorem 5.4.3 *When \mathcal{D} is a closed symmetric monoidal spectral category and the object set \mathcal{G} consists of cofibrant and fibrant objects the adjunction*

$$(-) \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G} : \text{mod-}\mathcal{E}(\mathcal{G}) \rightleftarrows \mathcal{D} : \underline{\text{Hom}}(\mathcal{G}, -)$$

is a spectral Quillen pair. If \mathcal{G} is a set of compact generators for \mathcal{D} then this Quillen pair is a spectral Quillen equivalence. If \mathcal{G} is closed under the monoidal product, then this pair is an adjunction of closed strong symmetric monoidal spectral model categories.

Proof The first two statements are [SS03b, Theorem 3.9.3]. So we must consider the case where \mathcal{G} is closed under the monoidal product and show that this adjunction is strong monoidal, this follows by [GS, Proposition 3.6] and we give an explicit proof below. The left adjoint is strong symmetric monoidal by the following series of isomorphisms, where we use the assumption that the collection of generators of \mathcal{D} form a symmetric monoidal Sp^Σ -category.

$$\begin{aligned} (M \square N) \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G} &= \int^{g \in \mathcal{G}} \int^{x, y \in \mathcal{G}} (\mathcal{E}(\mathcal{G})(g, x \wedge y) \wedge M(x) \wedge N(y)) \wedge g \\ &= \int^{x, y \in \mathcal{G}} M(x) \wedge N(y) \wedge \left(\int^{g \in \mathcal{G}} \mathcal{E}(\mathcal{G})(g, x \wedge y) \wedge g \right) \\ &\cong \int^{x, y \in \mathcal{G}} M(x) \wedge N(y) \wedge (x \wedge y) \\ &\cong \int^{x, y \in \mathcal{G}} (M(x) \wedge x) \wedge (N(y) \wedge y) \\ &\cong \int^{x \in \mathcal{G}} (M(x) \wedge x) \wedge \int^{y \in \mathcal{G}} (N(y) \wedge y) \\ &\cong \left(\int^{x \in \mathcal{G}} M(x) \wedge x \right) \wedge \left(\int^{y \in \mathcal{G}} N(y) \wedge y \right) \\ &= M \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G} \wedge N \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G} \end{aligned}$$

It is clear that the left adjoint preserves the Sp^Σ -algebra structure, that is, for a symmetric spectrum X ,

$$(X \wedge \mathcal{E}(\mathcal{G})(-, \mathbb{S})) \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G} \cong i_{\mathcal{D}}(X)$$

where $i_{\mathcal{D}}: Sp^{\Sigma} \rightarrow \mathcal{D}$ is the left adjoint of the Quillen pair giving \mathcal{D} a closed symmetric monoidal spectral category structure. We also need the following to be a weak equivalence in Sp^{Σ} .

$$\widehat{\mathcal{E}}(\mathcal{G})(-, \mathbb{S}) \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G} \rightarrow \mathcal{E}(\mathcal{G})(-, \mathbb{S}) \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G} \cong \mathbb{S}$$

We use the characterisation $\widehat{\mathcal{E}}(\mathcal{G})(-, \mathbb{S}) = \widehat{\mathbb{S}} \wedge \mathcal{E}(\mathcal{G})(-, \mathbb{S})$ and the result follows immediately. ■

In Chapter 9 we will need a slight alteration of this result to a context the unit object in \mathcal{G} is not cofibrant. We shall go through the changes needed in Theorem 9.1.2. It should be clear that the correct notion of compact for the above result when working in a G -equivariant setting is G -compactness.

Remark 5.4.4 The reader should note that the requirement that every object of \mathcal{G} is fibrant is essential to know that $\mathcal{E}(\mathcal{G})(g, g')$ has the correct homotopy type. It can be quite difficult to meet this requirement *and* have \mathcal{G} closed under the monoidal product since the smash product of a pair of fibrant objects in a model category is not usually fibrant. Equally, there is no reason to expect the monoidal product and fibrant replacement functor to be compatible. That is, the fibrant replacement functor is not usually a monoidal functor. The solution of [GS] and of our work is to work in a category with every object fibrant. This is the reason why we have constructed rational spectra as $S_{\mathbb{Q}}$ -modules in EKMM spectra. Similarly for the finite case we considered S_H -mod for each conjugacy class of subgroups (H) . Looking ahead to Part III, where we specialise to the group $O(2)$, we take our model for rational $SO(2)$ -spectra to be $\iota^* S_{\mathbb{Q}}$ -mod. Every object of this category is fibrant, hence so is every object of the associated skewed category.

Part III

The Continuous Dihedral Group

Chapter 6

Rational $O(2)$ -Equivariant Spectra

We now concentrate on the group $O(2)$. We apply our splitting result and state some of the homotopy level calculations on rational $O(2)$ -spectra from [Gre98b].

6.1 Basics

The closed subgroups of $O(2)$ are $O(2)$ itself, $SO(2)$, the finite dihedral groups and the finite cyclic groups. We will use the notation D_{2n}^h to represent the dihedral subgroup of order $2n$ containing h , an element of $O(2) \setminus SO(2)$. The conjugacy class of such subgroups for a fixed $n \geq 0$ will be written D_{2n} . We write C_n for the cyclic group of order n . For the rest of Part III, we let $W = O(2)/SO(2)$, the group of order 2. We define a **cyclic group** to be any subgroup of $SO(2)$, and a **dihedral group** to be any group of form D_{2n}^h or $O(2)$ itself. We take \mathcal{C} to be the family consisting of all cyclic subgroups of $O(2)$. The cofamily associated to \mathcal{C} consists of the dihedral groups, we call this set \mathcal{D} and we shall write $E\mathcal{D}$ for $E\tilde{\mathcal{C}}$.

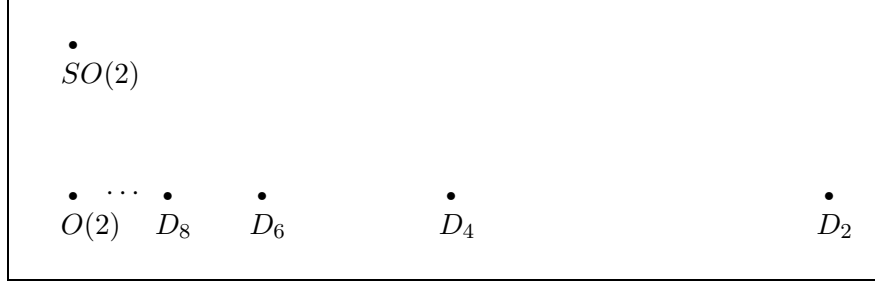
Lemma 6.1.1 *The family of cyclic subgroups, \mathcal{C} , is an idempotent family.*

Proof This follows from Lemma 3.4.10 since $SO(2)$ is the identity component of $O(2)$. We illustrate this result with Figure 6.1 below where we draw $\mathcal{F}O(2)/O(2)$. ■

Definition 6.1.2 The model category of **cyclic spectra** is $\mathcal{C}\mathcal{M}_{\mathbb{Q}}$ and the model category of **dihedral spectra** is $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$, which we write as $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$ (see Definition 3.4.2). In most cases we will no longer explicitly mention the fact that all categories are rationalised.

Theorem 6.1.3 *There is a strong monoidal Quillen equivalence*

$$\Delta : O(2)\mathcal{M}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{C}\mathcal{M}_{\mathbb{Q}} \times \mathcal{D}\mathcal{M}_{\mathbb{Q}} : \Pi.$$

Figure 6.1: $\mathcal{FO}(2)/O(2)$.

In particular, we have the following natural isomorphism for any G -spectra X and Y

$$[X, Y]_{\mathbb{Q}}^{O(2)} \cong [X \wedge E\mathcal{C}_+, Y \wedge E\mathcal{C}_+]_{\mathbb{Q}}^{O(2)} \oplus [X \wedge E\mathcal{D}, Y \wedge E\mathcal{D}]_{\mathbb{Q}}^{O(2)}.$$

Furthermore, we have a Quillen equivalence:

$$S_{\mathbb{Q}}\text{-mod} \xleftrightarrow{\quad} L_{E\mathcal{C}_+} S_{\mathbb{Q}}\text{-mod} \times L_{E\mathcal{D}} S_{\mathbb{Q}}\text{-mod}.$$

Proof This is an application of Theorem 3.4.14 and Theorem 3.3.2. ■

Denote the determinant representation of $O(2)$ by δ . This is a one dimensional real representation of $O(2)$. For $n > 0$, $n\delta$ is an n -dimensional real representation with $(n\delta)^H$ equal to 0 for $H \in \mathcal{D}$ and \mathbb{R}^n for $H \in \mathcal{C}$.

Lemma 6.1.4 *The universal space for the family \mathcal{C} is given by the universal space for the group W , that is, $E\mathcal{C} = EW$. A construction of universal space for the cofamily \mathcal{D} , $E\mathcal{D}$, is $S^{\infty\delta}$.*

Proof Consider EW as an $O(2)$ -space by letting $O(2)$ act via the quotient homomorphism $O(2) \rightarrow O(2)/SO(2) = W$. Then we note that $H \in \mathcal{C}$ acts trivially and $H \in \mathcal{D}$ acts through W , so that EW has the required universal property.

Using the inclusion of $n\delta$ into $(n+1)\delta$ we have $S^{n\delta} \rightarrow S^{(n+1)\delta}$ and the colimit of these maps is $S^{\infty\delta}$. Since each map is an inclusion, $(S^{\infty\delta})^H = \text{Colim}_n (S^{n\delta})^H$ and hence is equal to S^0 for $H \in \mathcal{D}$ and S^∞ for $H \in \mathcal{C}$. Since S^∞ is the infinite sphere it is weakly equivalent to a point. Hence $S^{\infty\delta}$ has the required universal property. ■

We compare our splitting above to [Gre98b, Proposition 3.1] and hence relate maps in $\text{Ho}\mathcal{CM}_{\mathbb{Q}}$ to maps of $\text{Ho}SO(2)\mathcal{M}_{\mathbb{Q}}$ with a W -action. In Chapter 8 we improve on this and obtain a model category version of this result. Note that since $SO(2)$ is a normal subgroup of $O(2)$, \mathcal{CM} is Quillen equivalent to $O(2)\mathcal{M}(SO(2))$, the $\mathcal{F}(SO(2))$ -model structure on $O(2)\mathcal{M}$, see Theorem 3.4.4.

Proposition 6.1.5 *The forgetful map induces a natural isomorphism $[X, Y]_{\mathcal{C}, \mathbb{Q}}^{O(2)} \cong \{[X, Y]_{\mathbb{Q}}^{SO(2)}\}^W$, where $\{[X, Y]_{\mathbb{Q}}^{SO(2)}\}^W$ is the set of maps in the homotopy category of rational $SO(2)$ -spectra with a homotopy action of W . Furthermore, $\text{Ho}\mathcal{CM}_{\mathbb{Q}}$ is equivalent to $\text{Ho}(SO(2)\mathcal{M}_{\mathbb{Q}})^W$, the homotopy category of rational $SO(2)$ -spectra with a homotopy action of W .*

Proof By Theorem 6.1.3 (see also Remark 3.2.6) it is clear that $[X, Y]_{\mathcal{C}, \mathbb{Q}}^{O(2)}$ is isomorphic to $e_{\mathcal{C}}[X, Y]_{\mathbb{Q}}^{O(2)}$.

The proof of Theorem 3.2.4 implies that this is isomorphic to $[E\mathcal{C}_+ \wedge X, Y]_{\mathbb{Q}}^{O(2)}$. Since $E\mathcal{C}_+ = EW_+$ we can apply [Gre98b, Proposition 3.1] to obtain the result. The final statement follows from [Gre98b, Corollary 3.2]. ■

Proposition 6.1.6 *Let $S_{\mathbb{Q}}\text{-mod}(\mathcal{C})$ denote the category of $S_{\mathbb{Q}}$ -modules in $O(2)\mathcal{M}$ with model structure created from the underlying category $O(2)\mathcal{M}(SO(2))$. Then the identity functor $S_{\mathbb{Q}}\text{-mod}(\mathcal{C}) \rightarrow L_{E\mathcal{C}_+}S_{\mathbb{Q}}\text{-mod}$ is the left adjoint of a Quillen equivalence.*

Proof As stated in [MM02, Chapter IV, Theorem 6.9] the identity functor from $O(2)\mathcal{M}(SO(2))$ to $\mathcal{C}\mathcal{M}$ is the left adjoint of a Quillen equivalence. This gives a Quillen pair between the module categories. By inspection these categories have the same weak equivalences, hence we have the Quillen equivalence as claimed. ■

We give the construction of the derived category of dihedral spectra from [Gre98b, Section 4]. For each $n \geq 1$ let V_n be a graded $\mathbb{Q}W$ -module and let $V_{\infty} = \text{Lim}_i \prod_{n \geq i} V_n$, a graded $\mathbb{Q}W$ -module. Let V_0 be a graded \mathbb{Q} -module and $\sigma: V_0 \rightarrow V_{\infty}$ be a W -map. Call such information a graded dihedral Mackey functor. A map of graded dihedral Mackey functors is a collection of W -maps $f_k: V_k \rightarrow V'_k$ for $k \geq 0$ such that the obvious square relating $\sigma: V_0 \rightarrow V_{\infty}$ to $\sigma': V'_0 \rightarrow V'_{\infty}$ commutes.

Proposition 6.1.7 *The homotopy category of dihedral spectra is equivalent to the category of graded dihedral Mackey functors.*

Proof This is a combination of [Gre98b, Summary 4.1 and Corollary 5.5]. ■

We have reduced our study of rational $O(2)$ -spectra to looking at cyclic spectra and dihedral spectra. For cyclic spectra we can use any of the model categories $\mathcal{C}\mathcal{M}_{\mathbb{Q}}$, $L_{E\mathcal{C}_+}S_{\mathbb{Q}}\text{-mod}$ or $S_{\mathbb{Q}}\text{-mod}(\mathcal{C})$, since these are all Quillen equivalent. For dihedral spectra we can use $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$ or $L_{E\mathcal{D}_+}S_{\mathbb{Q}}\text{-mod}$. The homotopy structure of dihedral spectra as described above is quite simple. I plan (in future work) to give a classification of dihedral spectra based on the results of Chapter 4.

The remainder of the thesis concentrates on the more subtle case of cyclic spectra. We have a description of the homotopy category of cyclic spectra, one can think of this description as saying that cyclic spectra are $SO(2)$ -spectra with some extra structure (a homotopy action of W). We wish to make this notion precise at the model category level so that we will have a better understanding of cyclic spectra. We first investigate the general idea in Chapter 7 and then apply this to the specific case of cyclic spectra in Chapter 8.

Chapter 7

Categories With Involution

In order to study cyclic $O(2)$ -spectra we consider the relation between $O(2)$ -spaces and $SO(2)$ -spaces (see Example 7.4.2). In the first section we have abstracted this relation to the notion of a category with involution (the analogue of $SO(2)$ -spaces) and its associated skewed category (the analogue of $O(2)$ -spaces). In the second section we have considered monoidal structures on these categories. In Section 7.3 we have given conditions for a model structure on the category with involution to pass to the skewed category. We have given some examples in Section 7.4 and we recommend that the impatient reader reads up to the definition of a skewed category and moves straight to these examples.

7.1 Categories with Involution

We give the basic definitions and constructions of categories with involution and their skewed categories. We then investigate the conditions necessary for a functor or an adjoint pair between categories with involution to pass to the skewed categories.

Definition 7.1.1 A **category with involution** (\mathcal{C}, σ) is a category \mathcal{C} with a functor $\sigma: \mathcal{C} \rightarrow \mathcal{C}$ such that $\sigma^2 = \text{Id}_{\mathcal{C}}$. We call such a functor σ an **involution**.

It follows, of course, that σ is both a left and a right adjoint. We could relax this definition by requiring that σ^2 is naturally isomorphic to $\text{Id}_{\mathcal{C}}$. In the following work we would then need to replace any use of the equality $\sigma^2 = \text{Id}_{\mathcal{C}}$ by the (specified and fixed) natural isomorphism $\sigma^2 \rightarrow \text{Id}_{\mathcal{C}}$. Note that our involution is a covariant self-functor of \mathcal{C} , this differs from some of the literature where an involution means a functor $\mathcal{C}^{op} \rightarrow \mathcal{C}$ which is self-inverse.

Definition 7.1.2 In a category with involution (\mathcal{C}, σ) , a **map of order two** is a map $f: A \rightarrow \sigma A$ such that $\sigma f = f^{-1}$.

Definition 7.1.3 Given a category with involution (\mathcal{C}, σ) , we define the associated **skewed category** $\sigma\#\mathcal{C}$ to be the category with objects $w: A \rightarrow \sigma A$ (also denoted

(A, w)), such that $w \circ \sigma w = \text{Id}_{\sigma A}$ (so w is a map of order two). A morphism of such objects is a commutative square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow w & & \downarrow w' \\ \sigma A & \xrightarrow{\sigma f} & \sigma B \end{array}$$

and we denote such a morphism by f .

Proposition 7.1.4 *Let (\mathcal{C}, τ) be a category with involution and assume that \mathcal{C} has coproducts, denoted \vee . Then there is an adjoint pair of functors*

$$\mathbb{D} : \mathcal{C} \rightleftarrows \sigma \# \mathcal{C} : \mathbb{P}.$$

The left adjoint is the free functor, and it acts on objects as $\mathbb{D}X = X \vee \sigma X \rightarrow \sigma X \vee X$ with twist w the interchange of factors map. The right adjoint is projection onto the first factor $\mathbb{P}(w : A \rightarrow \sigma A) = A$. These functors act on morphisms in the obvious way.

Proof See Lemma 7.3.3 for a more detailed construction of $\mathbb{D}X$. First let us understand what information is contained in the map

$$\begin{array}{ccc} X \vee \sigma X & \xrightarrow{(f,g)} & A \\ \downarrow w & & \downarrow w' \\ \sigma X \vee X & \xrightarrow{(\sigma f, \sigma g)} & \sigma A \end{array}$$

Thus we have the requirement $w' \circ (f, g) = (\sigma f, \sigma g) \circ w$, but $(\sigma f, \sigma g) \circ w = (\sigma g, \sigma f)$, hence $\sigma g = w' f$, so $g = \sigma(w' f)$. Note that $w' g = w' \circ \sigma w' \circ \sigma f = \sigma f$ since $w' \sigma w' = \text{Id}$. It follows that a map as above determines and is determined by a map $f : X \rightarrow A$ in \mathcal{C} , hence we have our adjunction. ■

Lemma 7.1.5 *Let (\mathcal{C}, τ) be a category with involution and assume that \mathcal{C} has products, denoted \prod . Then \mathbb{P} has a right adjoint.*

$$\mathbb{P} : \sigma \# \mathcal{C} \rightleftarrows \mathcal{C} : \mathbb{D}'$$

The right adjoint acts on objects as $\mathbb{D}'X = X \prod \sigma X \rightarrow \sigma X \prod X$, with twist w the interchange of factors map, it acts on morphisms in the obvious way.

Lemma 7.1.6 *Consider a general category \mathcal{C} , if \mathcal{C} has equalisers and coequalisers, then there we have a triple of functors $(\text{Orb}, \varepsilon, \text{Fix})$ arranged into adjoint pairs as below.*

$$\varepsilon : \mathcal{C} \rightleftarrows \text{Id} \# \mathcal{C} : \text{Fix} \quad \text{Orb} : \text{Id} \# \mathcal{C} \rightleftarrows \mathcal{C} : \varepsilon$$

Proof For an object A of \mathcal{C} , or $(B, u) \in \text{Id} \# \mathcal{C}$ we have the following definitions

$$\begin{aligned} \varepsilon A &= \text{Id} : A \rightarrow A \in \text{Id} \# \mathcal{C} \\ \text{Fix}(B, u) &= B^u = \text{Eq}(B \xrightarrow[u]{u} B) \in \mathcal{C} \\ \text{Orb}(B, u) &= B/u = \text{Coeq}(B \xrightarrow[u]{u} B) \in \mathcal{C} \end{aligned}$$

it is easy to see that these are adjoint pairs. ■

Remark 7.1.7 The category $\text{Id} \# \mathcal{C}$ is the category of C_2 -objects and C_2 -equivariant maps in \mathcal{C} . We have recovered the usual pair of triples: the forgetful functor $\text{Id} \# \mathcal{C} \rightarrow \mathcal{C}$ with its left and right adjoints and the trivial action functor $\mathcal{C} \rightarrow \text{Id} \# \mathcal{C}$ with its left and right adjoints.

Definition 7.1.8 An **involutive functor** $(F, \alpha): (\mathcal{C}, \sigma) \rightarrow (\mathcal{D}, \tau)$ consists of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\alpha: F\sigma \rightarrow \tau F$ such that $\tau\alpha \circ \alpha\sigma = \text{Id}_F$.

Lemma 7.1.9 An involutive functor $(F, \alpha): (\mathcal{C}, \sigma) \rightarrow (\mathcal{D}, \tau)$ passes to a functor $(\sigma, \tau) \# F: \sigma \# \mathcal{C} \rightarrow \tau \# \mathcal{D}$ which we call the **skewed functor**, we will often just call this functor F .

Proof We define $(\sigma, \tau) \# F$ on an object $w: A \rightarrow \sigma A$ to be the composite map $\alpha_A \circ Fw: FA \rightarrow \tau FA$. To see that this is an object of the category we draw the following commutative diagram.

$$\begin{array}{ccccccc}
 & & FA & & & & \\
 & & \downarrow Fw & \searrow & & & \\
 & = & F\sigma A & \xrightarrow{\alpha_A} & \tau FA & & \\
 & & \downarrow F\sigma w & & \downarrow \tau Fw & & \\
 FA & \xrightarrow{=} & F\sigma^2 A & \xrightarrow{\alpha_{\sigma A}} & \tau F\sigma A & \xrightarrow{\tau\alpha_A} & \tau^2 FA \xrightarrow{=} FA
 \end{array}$$

For a map $f: (w: A \rightarrow \sigma A) \rightarrow (w': A' \rightarrow \sigma A')$ in $\sigma \# \mathcal{C}$, we make the definition: $((\sigma, \tau) \# F)f = Ff$ (labelling maps by their first factor). This is a morphism of $\sigma \# \mathcal{C}$ by the naturality of α , as the diagram below demonstrates.

$$\begin{array}{ccc}
 FA & \xrightarrow{Ff} & FA' \\
 \downarrow Fw & & \downarrow Fw' \\
 F\sigma A & \xrightarrow{F\sigma f} & F\sigma A' \\
 \downarrow \alpha_A & & \downarrow \alpha_{A'} \\
 \tau FA & \xrightarrow{\tau Ff} & \tau FA'
 \end{array}$$

■

Definition 7.1.10 For a pair of involutive functors $(F, \alpha): (\mathcal{C}, \sigma) \rightarrow (\mathcal{D}, \tau)$ and $(G, \beta): (\mathcal{D}, \tau) \rightarrow (\mathcal{E}, \rho)$ we define the **composite involutive functor** to be

$$(G \circ F, \beta F \circ G\alpha): (\mathcal{C}, \sigma) \rightarrow (\mathcal{E}, \rho).$$

Lemma 7.1.11 The composite involutive functor of a composable pair of involutive functors is an involutive functor.

Proof By drawing a larger version of the diagram defining the skewed functor one can see that

$$\rho(\beta F \circ G\alpha) \circ (\beta F \circ G\alpha)\sigma = \text{Id}_{G \circ F}.$$

We draw the diagram below to explain the definition. Let $f: (A, w) \rightarrow (A', w')$ be a map in $\sigma \# \mathcal{C}$.

$$\begin{array}{ccc}
 GFA & \xrightarrow{GFf} & GFA' \\
 \downarrow GFw & & \downarrow GFw' \\
 GF\sigma A & \xrightarrow{GF\sigma f} & GF\sigma A' \\
 \downarrow G\alpha_A & & \downarrow G\alpha_{A'} \\
 G\tau FA & \xrightarrow{G\tau Ff} & G\tau FA' \\
 \downarrow \beta_{FA} & & \downarrow \beta_{FA'} \\
 \rho GFA & \xrightarrow{\rho GFf} & \rho GFA'
 \end{array}$$

■

Definition 7.1.12 An **involutive natural transformation** $\eta: (F, \alpha) \rightarrow (G, \beta)$ between involutory functors $(F, \alpha), (G, \beta): (\mathcal{C}, \sigma) \rightarrow (\mathcal{D}, \tau)$ is a natural transformation $\eta: F \rightarrow G$ such that $\tau\eta \circ \alpha = \beta \circ \eta\sigma$.

Lemma 7.1.13 An involutory natural transformation $\eta: (F, \alpha) \rightarrow (G, \beta)$ passes to a natural transformation between the skewed functors

$$\eta = (\sigma, \tau) \# \eta: (\sigma, \tau) \# F \rightarrow (\sigma, \tau) \# G$$

called the **skewed natural transformation**.

Proof All we need do is draw the diagram below, which will give the definition of $(\sigma, \tau) \# \eta$, explain the requirement $\tau\eta \circ \alpha = \beta \circ \eta\sigma$ and make it clear (through the naturality of each square) that this will be a natural transformation between the skewed functors.

$$\begin{array}{ccc}
 FA & \xrightarrow{\eta_A} & GA \\
 \downarrow Fw & & \downarrow Gw \\
 F\sigma A & \xrightarrow{\eta_{\sigma A}} & G\sigma A \\
 \downarrow \alpha_A & & \downarrow \beta_A \\
 \tau FA & \xrightarrow{\tau\eta_A} & \tau GA
 \end{array}$$

■

Definition 7.1.14 An **involutive adjunction** $((F, \alpha), (G, \beta), \eta, \varepsilon)$ between two categories with involution (\mathcal{C}, σ) and (\mathcal{D}, τ) is an adjunction $(F, G, \eta, \varepsilon)$ consisting of involutory functors $(F, \alpha): (\mathcal{C}, \sigma) \rightarrow (\mathcal{D}, \tau)$ and $(G, \beta): (\mathcal{D}, \tau) \rightarrow (\mathcal{C}, \sigma)$ and involutory natural transformations $\eta: \text{Id}_{\mathcal{C}} \rightarrow (GF, \beta F \circ G\alpha)$ and $\varepsilon: (FG, \alpha G \circ F\beta) \rightarrow \text{Id}_{\mathcal{D}}$.

Lemma 7.1.15 *An involutory adjunction $((F, \alpha), (G, \beta), \eta, \varepsilon)$ passes to an adjunction of the skewed categories. $((\sigma, \tau) \# F, (\tau, \sigma) \# G, (\sigma, \tau) \# \eta, (\tau, \sigma) \# \varepsilon)$. Furthermore, this gives the commutative square (see below) of adjoint functors below.*

$$\begin{array}{ccc}
 \sigma \# \mathcal{C} & \xrightleftharpoons[(\tau, \sigma) \# G]{(\sigma, \tau) \# F} & \tau \# \mathcal{D} \\
 \uparrow \mathbb{D} \quad \downarrow \mathbb{P} \quad \uparrow \mathbb{D}' & & \uparrow \mathbb{D} \quad \downarrow \mathbb{P} \quad \uparrow \mathbb{D}' \\
 \mathcal{C} & \xrightleftharpoons[G]{F} & \mathcal{D}
 \end{array}$$

Proof Since F and G are involutory they pass to the skewed categories. We have a unit and counit for the skewed categories by the assumption that η and ε are involutory. We must check the following pair of equations of natural transformations:

$$((\tau, \sigma) \# G \ (\tau, \sigma) \# \varepsilon) \circ ((\sigma, \tau) \# \eta \ (\tau, \sigma) \# G) = \text{Id}_{(\tau, \sigma) \# G}$$

$$((\sigma, \tau) \# F \ (\sigma, \tau) \# \eta) \circ ((\tau, \sigma) \# \varepsilon \ (\sigma, \tau) \# F) = \text{Id}_{(\sigma, \tau) \# F}$$

but this is immediate from our definitions and the fact that we started with an adjunction. By the word commutative, we are claiming that there are four natural isomorphisms as below.

$$\mathbb{D} \circ F \cong (\sigma, \tau) \# F \circ \mathbb{D} \quad \mathbb{P} \circ (\tau, \sigma) \# G \cong G \circ \mathbb{P}$$

$$\mathbb{P} \circ (\sigma, \tau) \# F \cong F \circ \mathbb{P} \quad (\tau, \sigma) \# G \circ \mathbb{D}' \cong \mathbb{D}' \circ G$$

It is easy to see that the isomorphisms containing \mathbb{P} exist. This is all that we need to check. ■

Remark 7.1.16 Consider a diagram of involutory adjunctions that commutes up to natural isomorphism. Then, in order to obtain a commuting diagram of skewed categories, one must require that the natural isomorphisms giving the commutativity are involutory.

The following lemma shows that in the case of an adjunction, one only has to check that one of the functors involved is involutory to deduce that the other functor is involutory.

Lemma 7.1.17 *Consider an adjunction $(F, G, \eta, \varepsilon)$ between two categories with involution (\mathcal{C}, σ) and (\mathcal{D}, τ) . Then there is a natural transformation $\alpha: F\sigma \rightarrow \tau F$ such that $\tau\alpha \circ \alpha\sigma = \text{Id}_F$ if and only if there is a natural transformation $\beta: G\tau \rightarrow \sigma G$ such that $\sigma\beta \circ \beta\tau = \text{Id}_G$.*

Proof Assume that α exists, this implies that $F\sigma$ and τF are naturally isomorphic functors. Hence their right adjoints σG and $G\tau$ are isomorphic, so we have a natural

transformation $\beta: G\tau \rightarrow \sigma G$. The following diagram must commute

$$\begin{array}{ccccc} \mathcal{D}(\tau^2 LX, Y) & \xrightarrow{=} & \mathcal{D}(LX, \tau^2 Y) & \xrightarrow{\phi} & \mathcal{C}(X, R\tau^2 Y) \\ \downarrow \tau\alpha\circ\alpha\sigma & & & & \downarrow \sigma\beta\circ\beta\tau \\ \mathcal{D}(L\sigma^2 X, Y) & \xrightarrow{\phi} & \mathcal{C}(\sigma^2 X, RY) & \xrightarrow{=} & \mathcal{C}(X, \sigma^2 RY) \end{array}$$

hence if $\tau\alpha\circ\alpha\sigma = \text{Id}_F$ then $\sigma\beta\circ\beta\tau = \text{Id}_G$. It is clear that the converse is also true. ■

7.2 Involutory Monoidal Categories

We consider the conditions necessary for a monoidal product on a category with involution to pass to a monoidal product on the skewed category.

Definition 7.2.1 An **involutory monoidal category** is a category with a closed monoidal product $(\otimes, \text{Hom}, \phi)$, a functor σ such that (\mathcal{C}, σ) is a category with involution and

- (i). a natural transformation $m: \sigma(-) \otimes \sigma(-) \rightarrow \sigma(- \otimes -)$ such that $\sigma m \circ m(\sigma \otimes \sigma) = \text{Id}_{- \otimes -}$,
- (ii). an isomorphism $i: \mathbb{I} \rightarrow \sigma \mathbb{I}$ such that $\sigma i \circ i = \text{Id}_{\mathbb{I}}$ (\mathbb{I} the unit of \otimes).

That is, we require σ to be a strong monoidal functor (σ, m, i) , such that m behaves in a similar way to an involutory natural transformation and i is a map of order two. If \mathcal{C} is a symmetric monoidal category and σ is a symmetric monoidal functor then (\mathcal{C}, σ) is an **involutory symmetric monoidal category**.

Theorem 7.2.2 *The skewed category of an involutory symmetric monoidal category is a symmetric monoidal category.*

Proof Let (\mathcal{C}, σ) be a the category with involution, with closed monoidal structure $(\otimes, \text{Hom}, \phi, m, i)$ such that \mathcal{C} is an involutory monoidal category. We will use $(\otimes, \text{Hom}, \phi)$ to denote the monoidal product on the skewed category as decorating these symbols further would be horrific. We begin by defining the action of the functor \otimes on a pair of objects $u: A \rightarrow \sigma A$ and $v: B \rightarrow \sigma B$ to be

$$m_{A,B} \circ u \otimes v: A \otimes B \rightarrow \sigma A \otimes \sigma B \rightarrow \sigma(A \otimes B).$$

This is an object of the skewed category since m is an involutory natural transformation. The action on maps is obvious from this definition. The unit is the object $i: \mathbb{I} \rightarrow \sigma \mathbb{I}$, since i is a map of order two, this is also an object of the category.

Now one must check that $i: \mathbb{I} \rightarrow \sigma \mathbb{I}$ defines a unit, that the product as above is associative and that certain coherence diagrams are satisfied see either [Mac71, VII] or [Hov99, Chapter 4]. This is all routine and follows from our assumption that σ is strong monoidal, most coherence diagrams are easy to check, but one can use the

‘coherence theorem’ of [Mac71, VII] to note that all the diagrams that we are checking commute. Thus we have a monoidal product on $\sigma\#\mathcal{C}$ and it is clear that this product is symmetric provided \otimes is symmetric on \mathcal{C} and σ is a strong symmetric monoidal functor.

Now we move to showing that this monoidal structure is closed, so we construct an internal function object. Let B be an object of \mathcal{C} . The functors $-\otimes\sigma B$ and $\sigma(\sigma(-)\otimes B)$ are naturally isomorphic, hence (by Lemma 7.1.17) there is a canonical natural isomorphism of order two $\text{Hom}(\sigma B, -) \rightarrow \sigma \text{Hom}(B, \sigma-)$. A small amount of adjustment will give $f_{A,B}: \text{Hom}(\sigma A, \sigma B) \rightarrow \sigma \text{Hom}(A, B)$ and it follows that $\sigma f \circ f(\sigma, \sigma) = \text{Id}$, that is, f is an involutory natural transformation in the same way that m is.

Alternatively one can construct this natural transformation via the following diagram (note that since $\sigma^2 = \text{Id}$, a general object of \mathcal{C} can be written as σA for suitable A). Take $A = \sigma \text{Hom}(\sigma B, \sigma C)$ and follow the identity map round the diagram to obtain the natural transformation $f: \text{Hom}(\sigma-, \sigma-) \rightarrow \sigma \text{Hom}(-, -)$.

$$\begin{array}{ccccc} \text{Hom}(\sigma A, \text{Hom}(\sigma B, \sigma C)) & \xrightarrow{\phi_{\sigma A, \sigma B, \sigma C}^{-1}} & \text{Hom}(\sigma A \otimes \sigma B, \sigma C) & \xrightarrow{(\sigma m_{\sigma A, \sigma B})^*} & \text{Hom}(\sigma(A \otimes B), \sigma C) \\ & & & & \downarrow \sigma \\ \text{Hom}(\sigma A, \sigma \text{Hom}(B, C)) & \xleftarrow{\sigma} & \text{Hom}(A, \text{Hom}(B, C)) & \xleftarrow{\phi_{A, B, C}} & \text{Hom}(A \otimes B, C) \end{array}$$

Let (A, u) , (B, v) and (C, w) be objects of the skewed category, we define the object $\text{Hom}((A, u), (B, v))$ to be

$$f_{A,B} \circ \text{Hom}(\sigma u, v): \text{Hom}(A, B) \rightarrow \text{Hom}(\sigma A, \sigma B) \rightarrow \sigma \text{Hom}(A, B).$$

We must prove that maps $(A, u) \otimes (B, v) \rightarrow (C, w)$ are in natural bijection with maps $(A, u) \rightarrow \text{Hom}((B, v), (C, w))$. Consider the following pair of triangles, it follows from the construction of the natural transformation f above that the left hand triangle commutes if and only if the right hand triangle commutes.

$$\begin{array}{ccc} \sigma A \otimes \sigma B & \xrightarrow{g} & \sigma C \\ m_{A,B} \downarrow & \nearrow \sigma k & \\ \sigma(A \otimes B) & & \end{array} \quad \begin{array}{ccc} \sigma A & \xrightarrow{\phi g} & \text{Hom}(\sigma B, \sigma C) \\ \searrow \sigma \phi k & & \downarrow f_{B,C} \\ & & \sigma \text{Hom}(B, C) \end{array}$$

We draw another picture showing three squares. If any one of these commutes, so do the other two.

$$\begin{array}{ccccc} A \otimes B & \xrightarrow{k} & C & & A & \xrightarrow{\phi k} & \text{Hom}(B, C) & & A & \xrightarrow{\phi k} & \text{Hom}(B, C) \\ \downarrow u \otimes v & & \downarrow w & & \downarrow u & & \downarrow \text{Hom}(\text{Id}, w) & & \downarrow u & & \downarrow \text{Hom}(\sigma v, w) \\ \sigma A \otimes \sigma B & \xrightarrow{g} & \sigma C & & \sigma A & \xrightarrow{\phi g} & \text{Hom}(\sigma B, \sigma C) & & \sigma A & \xrightarrow{\phi g} & \text{Hom}(\sigma B, \sigma C) \\ & & & & \uparrow \text{Hom}(v, \text{Id}) & & & & & & \end{array}$$

Fix k , then by the above $g = w \circ k \circ (\sigma u \otimes \sigma v)$. We combine these two collections of diagrams (and remove g from the result) to obtain the pair of diagrams below. The left hand diagram commutes if and only if the right hand diagram commutes. Thus we have our adjunction on the skewed category.

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{k} & C \\
 \downarrow u \otimes v & & \downarrow w \\
 \sigma A \otimes \sigma B & & \sigma C \\
 \downarrow m_{A,B} & & \\
 \sigma(A \otimes B) & \xrightarrow{\sigma k} & \sigma C
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\phi k} & \text{Hom}(B, C) \\
 \downarrow u & & \downarrow \text{Hom}(\sigma v, w) \\
 & & \text{Hom}(\sigma B, \sigma C) \\
 & & \downarrow f_{B,C} \\
 \sigma A & \xrightarrow{\sigma \phi k} & \sigma \text{Hom}(B, C)
 \end{array}$$

■

One can remove the assumption that \otimes is symmetric. The above proof would suffice to show that one obtains a monoidal skewed category, but one would have to take care over the fact that there are now two (possibly different) right adjoints to \otimes (these are Hom_l and Hom_r from Definition 1.1.10).

Definition 7.2.3 An **involutive monoidal adjunction** is an involutive adjunction $(L, \alpha) : (\mathcal{C}, \sigma) \rightleftarrows (\mathcal{D}, \tau) : (R, \beta)$ such that L is op-monoidal and R is monoidal and β is a monoidal natural transformation $R\tau \rightarrow \sigma R$. An **involutive symmetric monoidal adjunction** between involutive symmetric monoidal categories is an involutive monoidal adjunction that is a symmetric monoidal adjunction.

Lemma 7.2.4 *An involutive monoidal adjunction passes to a monoidal adjunction on the skewed categories. If this adjunction is strong monoidal or symmetric monoidal then so is the adjunction on the skewed category.*

Proof Take an involutive monoidal adjunction $(L, \alpha) : (\mathcal{C}, \sigma) \rightleftarrows (\mathcal{D}, \tau) : (R, \beta)$. The assumptions on β are the same as requiring that the diagrams below commute.

$$\begin{array}{ccc}
 \mathbb{I}_{\mathcal{C}} & \xrightarrow{\nu} & R\mathbb{I}_{\mathcal{C}} \\
 \downarrow i_{\mathcal{C}} & & \downarrow Ri_{\mathcal{D}} \\
 & & R\tau\mathbb{I}_{\mathcal{C}} \\
 & & \downarrow \beta \\
 \sigma\mathbb{I}_{\mathcal{C}} & \xrightarrow{\sigma\nu} & \sigma R\mathbb{I}_{\mathcal{C}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 R\tau d \otimes R\tau d' & \longrightarrow & R\tau(d \otimes d') \\
 \downarrow \beta_d \otimes \beta_{d'} & & \downarrow \beta_{d \otimes d'} \\
 \sigma R d \otimes \sigma R d' & \longrightarrow & \sigma R(d \otimes d')
 \end{array}$$

Thus R passes to a monoidal functor on the skewed category, hence the skewed adjunction is monoidal. The statement regarding strong monoidal adjunctions is obvious. For the symmetric monoidal statement it is easy to check that if L and R are symmetric then so are $\sigma\#L$ and $\tau\#R$. ■

Definition 7.2.5 Let (\mathcal{C}, σ) be an involutive (symmetric) monoidal model category. Then (\mathcal{C}, σ) is an **involutive (symmetric) closed ν -algebra** if there is a strong

(symmetric) monoidal involutory adjunction $i_{\mathcal{C}} : \nu \rightleftarrows \mathcal{C} : j_{\mathcal{C}}$ with respect to the involution Id_{ν} on ν .

Lemma 7.2.6 *The skewed category of an involutory (symmetric) closed ν -algebra is a closed (symmetric) $\text{Id} \# \nu$ -algebra. Furthermore it is a closed ν -algebra.*

Proof The first statement of the lemma is simply that an involutory adjunction passes to an adjunction of the skewed categories. The second statement follows from composing (i, j) with the strong monoidal adjunction $(\varepsilon, \text{Fix})$. ■

7.3 Involutary Model Categories

The next logical step in developing categories with involution is to consider model structures. We provide criteria for model structures and Quillen functors on involutory model categories to pass to the skewed categories.

Definition 7.3.1 An **involutory model category** (M, σ) is a cofibrantly generated model category M with left Quillen functor $\sigma : M \rightarrow M$ such that $\sigma^2 = \text{Id}_M$. That is, (M, σ) is a category with involution.

It follows, of course, that σ is also a right Quillen functor and that σ preserves all weak equivalences (since a weak equivalence is a composite of an acyclic cofibration followed by an acyclic fibration).

Definition 7.3.2 An **involutory Quillen functor** is an involutory functor that is a Quillen functor.

Lemma 7.3.3 *The category $\sigma \# M$ has all small limits and colimits.*

Proof Take a diagram D (i.e. a small category) and a functor $F : D \rightarrow \sigma \# M$. We can form $\text{Colim}_D \mathbb{P}F$ (since M is bicomplete) and since σ is a left adjoint we have a canonical isomorphism $\sigma \text{Colim}_D \mathbb{P}F \cong \text{Colim}_D \sigma \mathbb{P}F$. We also have a map $\text{Colim}_D w : \text{Colim}_D \mathbb{P}F \rightarrow \text{Colim}_D \sigma \mathbb{P}F$ we combine these maps to give an object of the skewed category in the diagram below.

$$\begin{array}{ccccccc}
 & & \text{Colim}_D \mathbb{P}F & & & & \\
 & \swarrow & \downarrow \text{Colim}_D w & \searrow & & & \\
 & & \text{Colim}_D \sigma \mathbb{P}F & \xrightarrow{\quad} & \sigma \text{Colim}_D \mathbb{P}F & & \\
 & \swarrow & \downarrow \text{Colim}_D \sigma w & & \downarrow \sigma \text{Colim}_D w & \searrow & \\
 \text{Colim}_D \mathbb{P}F & \xrightarrow{=} & \text{Colim}_D \sigma^2 \mathbb{P}F & \xrightarrow{\quad} & \sigma \text{Colim}_D \sigma \mathbb{P}F & \xrightarrow{\quad} & \sigma^2 \text{Colim}_D \mathbb{P}F \\
 & & & & & & \downarrow = \\
 & & & & & & \text{Colim}_D \mathbb{P}F
 \end{array}$$

The composite $\text{Colim}_D \mathbb{P}F \rightarrow \sigma^2 \text{Colim}_D \mathbb{P}F$ is the identity since there is a unique isomorphism between any two colimits of a diagram. The case for a limit is identical since σ is also a right adjoint. ■

Definition 7.3.4 Let (M, σ) be an involutory model category, we define a **weak equivalence** (respectively **fibration**) of $\sigma \# M$ to be a map f such that $\mathbb{P}f$ is a weak equivalence (respectively fibration).

Proposition 7.3.5 *These weak equivalences and fibrations define a model structure on $\sigma \# M$ and we call this category and model structure the **skewed model category**.*

Proof All that is required is to check that the lifting lemma (below) applies in the case of

$$\mathbb{D} : M \rightleftarrows \sigma \# M : \mathbb{P}.$$

We have already shown that \mathbb{P} is a left adjoint and so preserves all colimits. Let I and J denote the generating cofibrations and acyclic cofibrations of M . Now we must check that every relative $\mathbb{D}J$ -cell complex is a weak equivalence, Take $k : A \rightarrow B$ a map in J , then

$$\mathbb{P}\mathbb{D}k = k \vee \sigma k : A \vee \sigma A \rightarrow B \vee \sigma B$$

and since σ is a left Quillen functor (on M), it follows that $\mathbb{P}\mathbb{D}k$ is an acyclic cofibration in M . A relative $\mathbb{D}J$ -cell complex in $\sigma \# M$ is a transfinite composition of pushouts of $\mathbb{D}J$. Since \mathbb{P} preserves these constructions (it is a left adjoint) and the set of acyclic cofibrations (of M) are closed under these operations ([Hov99, proof of 2.2.10]), the result follows. ■

The original reference for the lifting lemma is of course [Qui67, II.4]. The lemma below is a variation on [Hir03, Theorem 11.3.2] where we assume that the right adjoint preserves filtered colimits so that the required smallness conditions hold.

Lemma 7.3.6 (Lifting Lemma) *Let $F : M \rightleftarrows N : G$ be an adjoint pair of functors (F is the left adjoint) with M a cofibrantly generated model category. Let I be the generating cofibrations and J the generating acyclic cofibrations for M . Assume that N has all small colimits and limits. Define a map f in N to be a weak equivalence or a fibration if and only if Gf is so. Define the cofibrations of N to be those maps in N with the correct lifting property. Then this construction defines a cofibrantly generated model structure on N provided:*

- (i). G preserves filtered colimits,
- (ii). every relative FJ -cell complex is a weak equivalence.

The sets FI and FJ are the generating cofibrations and generating acyclic cofibrations for N .

Lemma 7.3.7 *Let (M, σ) be a model category with involution, then $\mathbb{P} : \sigma \# M \rightarrow M$ preserves cofibrations and hence is a left Quillen functor.*

Proof The right adjoint \mathbb{D}' preserves fibrations and acyclic fibrations. ■

Definition 7.3.8 An **involutive Quillen pair** is an involutive adjunction that is also a Quillen pair.

Lemma 7.3.9 *An involutive Quillen pair between model categories with involution, passes to a Quillen pair between the skewed model categories.*

Proof Let $(F, \alpha) : (M, \sigma) \rightleftarrows (N, \tau) : (G, \beta)$ be an involutive Quillen pair. We will show that $(\tau, \sigma) \# G$ is a right Quillen functor. By Lemma 7.1.15 we have the equation $\mathbb{P}_M \circ (\tau, \sigma) \# G = G \circ \mathbb{P}_N$. Take a map f in $\tau \# N$, if f is a fibration or acyclic fibration then so is $G(\mathbb{P}_N(f))$. Hence $(\tau, \sigma) \# G(f)$ is a fibration or acyclic fibration by the definition of the model structure on $\sigma \# M$. ■

Definition 7.3.10 An **involutive Quillen equivalence** is an involutive Quillen pair that is also a Quillen equivalence.

Proposition 7.3.11 *An involutive Quillen equivalence passes to Quillen equivalence of the skewed model categories.*

Proof Take an involutive Quillen equivalence $(F, \alpha) : (M, \sigma) \rightleftarrows (N, \tau) : (G, \beta)$ then consider a cofibrant $c \rightarrow \sigma c$ in $\sigma \# M$ and a fibrant $d \rightarrow \tau d$ in $\tau \# N$. Then c is cofibrant in M and d is fibrant in N so a map $Fc \rightarrow d$ is a weak equivalence if and only if $c \rightarrow Gd$ is a weak equivalence. But this is precisely the statement that a map $(Fc \rightarrow \tau Fc) \rightarrow (d \rightarrow \tau d)$ is a weak equivalence if and only if $(c \rightarrow \sigma c) \rightarrow (Gd \rightarrow \sigma Gd)$ is a weak equivalence. ■

Definition 7.3.12 An **involutive (symmetric) monoidal model category** is a category with involution (M, σ) such that:

- (i). M is a (symmetric) monoidal model category,
- (ii). (M, σ) is an involutive model category,
- (iii). (M, σ) is an involutive (symmetric) monoidal category.

Lemma 7.3.13 *If (M, σ) is an involutive (symmetric) monoidal model category then $\sigma \# M$ is a (symmetric) monoidal model category. Furthermore, if M satisfies the monoid axiom, so does $\sigma \# M$.*

Proof By the machinery above we know that $\sigma \# M$ is a (symmetric) monoidal category. Hence we must show that this is a monoidal model category. To do so we use the following alternative form of the pushout product axiom (see [Hov99, Lemma 4.2.2]). Take $f : (C, u) \rightarrow (D, v)$ a cofibration in $\sigma \# M$ and $g : (X, r) \rightarrow (Y, s)$ a fibration. Then we must prove that the induced map

$$\alpha : \text{Hom}((D, v), (X, r)) \rightarrow \text{Hom}((D, v), (Y, s)) \prod_{\text{Hom}((C, u), (Y, s))} \text{Hom}((C, u), (X, r))$$

is a fibration that is acyclic whenever f or g is. Since the codomain of α is a pullback we can apply \mathbb{P} and obtain a pullback diagram in M , we draw this below..

$$\begin{array}{ccccc}
 & \text{Hom}(D, X) & & & \\
 & \searrow & \searrow \mathbb{P}\alpha & \searrow & \\
 & & P & \xrightarrow{\quad} & \text{Hom}(C, X) \\
 & \searrow & \downarrow & & \downarrow \\
 & & \text{Hom}(D, Y) & \xrightarrow{\quad} & \text{Hom}(C, Y)
 \end{array}$$

By the definition of the model structure on $\sigma\#M$ we must check that $\mathbb{P}\alpha$ is a fibration that is acyclic whenever $\mathbb{P}f$ or $\mathbb{P}g$ is. Since $\mathbb{P}f$ is a cofibration and $\mathbb{P}g$ is a fibration the result follows by the pushout product axiom for M .

We must also prove a result concerning the cofibrant replacement of \mathbb{I} . Let \widehat{c} be cofibrant replacement in $\sigma\#M$, the diagram

$$\begin{array}{ccccc}
 * & \xrightarrow{\quad} & \widehat{c}\mathbb{I} & \xrightarrow{\sim} & \mathbb{I} \\
 \downarrow & & \downarrow \widehat{c}i & & \downarrow i \\
 * & \xrightarrow{\quad} & \sigma\widehat{c}\mathbb{I} & \xrightarrow{\sim} & \sigma\mathbb{I}
 \end{array}$$

gives a cofibrant replacement of the unit in $\sigma\#M$. Thus $* \rightarrow \widehat{c}\mathbb{I} \rightarrow \mathbb{I}$ is a cofibrant replacement of the unit in M . Let (X, r) be a cofibrant object of $\sigma\#M$, then $X = \mathbb{P}(X, r)$ is cofibrant in M . Since M is a monoidal model category, $\widehat{c}\mathbb{I} \otimes X \rightarrow \mathbb{I} \otimes X$ is a weak equivalence. Thus

$$(\widehat{c}\mathbb{I}, \widehat{c}i) \otimes (X, r) \rightarrow (\mathbb{I}, i) \otimes (X, r)$$

is a weak equivalence in $\sigma\#M$. The statement regarding the monoid axiom holds by an equally straightforward argument using the fact that \mathbb{P} preserves cofibrations. ■

Definition 7.3.14 An **involutive (symmetric) monoidal Quillen pair** is an involutory (symmetric) monoidal adjunction that is also a monoidal Quillen pair.

Proposition 7.3.15 *An involutory monoidal Quillen pair induces a monoidal Quillen pair on the skewed categories. If the involutory pair is strong monoidal or symmetric then so is the skewed adjunction.*

Proof Let $(F, \alpha) : (M, \sigma) \rightleftarrows (N, \tau) : (G, \beta)$ be our involutory monoidal pair. By Lemma 7.2.4 (F, G) passes to a monoidal adjunction on the skewed category. This skewed pair will be strong monoidal or symmetric when (F, G) is. All that remains to check is that two technical conditions of a monoidal Quillen pair hold for the skewed adjunction. That is, we must check that if (X, u) and (Y, v) are cofibrant objects of $\sigma\#M$ then the map

$$F((X, u) \otimes (Y, v)) \rightarrow F(X, u) \otimes F(Y, v)$$

is a weak equivalence. Let (\mathbb{I}_M, i_M) be the unit of the skewed category $\sigma \# M$. We must also check that the map below is a weak equivalence for a cofibrant replacement $(\widehat{c}\mathbb{I}_M, \widehat{c}i_M)$ of (\mathbb{I}_M, i_M) .

$$F(\widehat{c}\mathbb{I}_M, \widehat{c}i_M) \rightarrow F(\mathbb{I}_M, i_M) \rightarrow (\mathbb{I}_N, i_N)$$

Both of these conditions hold since (F, G) is a monoidal Quillen pair and the underlying object X of a cofibrant object $(X, u) \in \sigma \# M$ is cofibrant in M . ■

Proposition 7.3.16 *Take (M, σ) an involutory symmetric monoidal model category satisfying the monoid axiom and let $\alpha: R \rightarrow \sigma R$ a commutative ring object in $\sigma \# M$. Then $(R\text{-mod}, \alpha^* \sigma)$ is an involutory symmetric monoidal model category and is equal to $(R, \alpha)\text{-mod}$, the category of (R, α) -modules in $\sigma \# M$.*

Proof An object of $(R\text{-mod}, \alpha^* \sigma)$ is an R -module X and an R -module map of order two $w: X \rightarrow \alpha^* \sigma X$. It is important to recall that α^* only affects the R -module structure of an object of $R\text{-mod}$, the underlying object of X is unchanged. Thus we have the diagram

$$\begin{array}{ccc} R \wedge X & \xrightarrow{\nu} & X \\ \downarrow \text{Id} \wedge w & & \downarrow w \\ R \wedge \alpha^* \sigma X & & \\ \downarrow \alpha \wedge \text{Id} & & \\ \sigma R \wedge \sigma x & \xrightarrow{m} \sigma(R \wedge X) \xrightarrow{\sigma \nu} & \alpha^* \sigma X \end{array}$$

where we note that $\sigma \nu \circ m \circ \alpha \wedge \text{Id}$ is the R -module action on $\alpha^* \sigma X$. It is clear that this precisely the requirement that X is an (R, α) -module in $\sigma \# M$. Equally one can see that the morphisms in these two categories agree. The (acyclic) fibrations for each of the above categories come from the model structure on M and hence are the same. Since R is commutative, $R\text{-mod}$ is a monoidal model category satisfying the monoid axiom, hence so is $(R\text{-mod}, \alpha^* \sigma)$. ■

Remark 7.3.17 We have constructed all the above for the case of a functor σ such that $\sigma^2 = \text{Id}$. So an involutory category \mathcal{C} is a category with an action of C_2 . The above can be generalised to a category with an action of a group G . That is a category \mathcal{C} , with a group homomorphism $G \rightarrow \text{Aut}(\mathcal{C})$ (invertible functors $\mathcal{C} \rightarrow \mathcal{C}$). Extending all of the above to general G should be formal, though the notation would have to be revised.

7.4 Examples

Example 7.4.1 Our first example of a category with involution is a category \mathcal{C} with the identity functor. As described in Remark 7.1.7 the skewed category is the category of C_2 -objects and maps in \mathcal{C} . Thus $\text{Id} \# \mathcal{C}$ is the category of functors C_2 to \mathcal{C} , where C_2 is the one-object category of a group. Moreover, if \mathcal{C} has a monoidal product, then (\mathcal{C}, Id) is clearly an involutory monoidal category. The monoidal product on $\text{Id} \# \mathcal{C}$ is

then the usual product of C_2 -objects in \mathcal{C} . Similarly any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ passes to the skewed categories $F: \text{Id} \# \mathcal{C} \rightarrow \text{Id} \# \mathcal{D}$ since F preserves C_2 -objects and C_2 -maps.

When one considers model structures there is a subtlety to consider, which will reoccur in Example 7.4.2. A map $f: (X, u) \rightarrow (Y, v)$ is a weak equivalence or fibration if and only if $f: X \rightarrow Y$ is so in \mathcal{C} . If \mathcal{C} is the category of topological spaces then the usual model structure for C_2 -spaces has weak equivalences and fibrations those maps $f: X \rightarrow Y$ such that f and $f^{C_2}: X^{C_2} \rightarrow Y^{C_2}$ are weak equivalences or fibrations in \mathcal{T}_* . The model structure we have constructed on $\text{Id} \# \mathcal{T}_*$ is the \mathcal{T} -model structure on C_2 -spaces as mentioned on [MM02, Page 70], for \mathcal{T} the family consisting of the identity subgroup. This is also known as the model category of **free C_2 -spaces**.

Example 7.4.2 This should be regarded as our motivating example: it describes $O(2)$ -spaces as $SO(2)$ -spaces with extra structure. For $t \in SO(2)$ and $k \in O(2) \setminus SO(2)$ we have the equation $ktkt = 1$ in $O(2)$, (t is a rotation and k is a reflection). Conjugation by k considered as an automorphism of $O(2)$ restricts to the inversion automorphism j of $SO(2)$, $j(t) = t^{-1}$. For a general group homomorphism $f: G \rightarrow H$, we can pull X , an H -space, back to a G -space by $g * x = f(g) \cdot x$. Since this is a contravariant construction, we use an upper asterisk and call this G -space f^*X . The underlying set of f^*X is the same and for a G -map $g: X \rightarrow Y$, the underlying set map of f^*g is the same as that for g . Thus the group homomorphism j gives j^* , an involution on $SO(2)$ -spaces. We thus have the skewed category $j^* \# SO(2) \mathcal{T}_*$, this is an involutory symmetric monoidal model category. We now see how this category relates to $O(2)$ -spaces.

Take X an $O(2)$ -space then we can consider ι^*X , the restriction of X to an $SO(2)$ -space along $\iota: SO(2) \rightarrow O(2)$. For the rest of this example h will be a fixed reflection. Since X is an $O(2)$ -space we have a map $h: \iota^*X \rightarrow j^*\iota^*X$. We show that this is an $SO(2)$ map, take $t \in SO(2)$ and let \cdot be the $SO(2)$ action on ι^*X and $*$ the action of $SO(2)$ on $j^*\iota^*X$. We have the equation

$$t * (h \cdot x) = t^{-1} \cdot (h \cdot x) = (t^{-1}h) \cdot x = (ht) \cdot x = h \cdot (t \cdot x).$$

We can also consider h as a map $j^*\iota^*X \rightarrow \iota^*X$ and it is clear that h is a map of order two. Hence for an $O(2)$ -space X , (ι^*X, h) is an object of $j^* \# SO(2) \mathcal{T}_*$. Thus we have a functor $I: O(2) \mathcal{T}_* \rightarrow j^* \# SO(2) \mathcal{T}_*$.

It should be clear that an object of $j^* \# SO(2) \mathcal{T}_*$ defines an $O(2)$ -space. Let (X, w) be an object of the skewed category, then we let $O(2)$ act on X by defining $h: X \rightarrow X$ to be the map w . This defines a functor $C: j^* \# SO(2) \mathcal{T}_* \rightarrow O(2) \mathcal{T}_*$. It is easy to see that (C, I) are an adjoint equivalence between $j^* \# SO(2) \mathcal{T}_*$ and $O(2) \mathcal{T}_*$, furthermore C is a strong monoidal functor. The choice of h is unimportant, as one would expect since any two reflections are conjugate by a rotation. Different choices of h will give different adjoint pairs between $O(2)$ -spaces and $j^* \# SO(2) \mathcal{T}_*$. These can be compared using the change of groups functor on $O(2)$ -spaces by considering conjugation by a rotation as a group homomorphism of $O(2)$. Note that for a $SO(2)$ -space Y there won't be usually be an $SO(2)$ -map of order two $Y \rightarrow j^*Y$.

Now we turn to model structures, we can put a model structure on $O(2)$ -spaces where a map f is a weak equivalence or fibration if and only if ι^*f is so in $SO(2)$ -spaces. This

model structure is the \mathcal{C} -model structure on $O(2)\mathcal{T}_*$ as mentioned on [MM02, Page 70], for \mathcal{C} the family of subgroups of $SO(2)$. If we call this model structure $\mathcal{C}\mathcal{T}_*$, then we can summarise this example in the statement: there is a strong symmetric monoidal Quillen equivalence between $j^*\#SO(2)\mathcal{T}_*$ and $\mathcal{C}\mathcal{T}_*$.

Example 7.4.3 We now consider a well-known algebraic example. We take the following definition from [MR01, 5.4]: for a ring R and a group G with G acting on R by $r \rightarrow r^g$, the **skew group ring**, $R\#G$, is the free R -module with G as a basis and multiplication defined as $(hr)(gs) = (hg)(r^g s)$. Note that this ring is not commutative.

Let R be a commutative ring and $w: R \rightarrow R$ a ring map such that $w^2 = \text{Id}_R$ and we obtain $R\#W$. We also have an involution w^* (pullback along w) on the category of R -modules. It is easy to see that the category of $R\#W$ -modules and $w^*\#(R\text{-mod})$ are isomorphic. One can show that $R\#W$ is a co-associative, co-commutative co-algebra. Furthermore the co-product (Δ) is compatible with the ring multiplication and unit. We use this to define a monoidal product, for $R\#W$ -modules M and N their product is $M \otimes_R N$ with $R\#W$ -module structure given by the composite

$$\begin{aligned} R\#W \otimes_R (M \otimes_R N) &\xrightarrow{\Delta} (R\#W \otimes_R R\#W) \otimes_R (M \otimes_R N) \\ &\xrightarrow{T} (R\#W \otimes_R M) \otimes_R (R\#W \otimes_R N) \\ &\rightarrow M \otimes_R N. \end{aligned}$$

This monoidal product on $R\#W\text{-mod}$ corresponds precisely to the monoidal product on $w^*\#(R\text{-mod})$. We have model categories of $dgR\#W$ -modules and dgR -modules, using the projective model structure. All of the previous material of this example still applies and the isomorphism of categories between $dgR\#W$ -modules and $w^*\#(dgR\text{-mod})$ is an isomorphism of (monoidal) model structures. We must mention that [SS00, Page 504] briefly considers skew group rings, but doesn't mention the monoidal structure that we have considered and uses a different kind of model structure. This example is the reason why we have chosen the notation $\sigma\#\mathcal{C}$ and the name skewed category.

Example 7.4.4 In [Ati66] a cohomology theory KR is defined, it relates to vector bundles of the following form. Let X be a space with $\mathbb{Z}/2$ -action ($u: X \rightarrow X$). Consider a complex vector bundle E over X with a map of order two $f: E \rightarrow u^*\bar{E}$, where \bar{E} is the conjugate bundle of E . Then for suitably nice spaces X , $KR(X)$ is the Grothendieck group of isomorphism classes of vector bundles over X with a map of order two $f: E \rightarrow u^*\bar{E}$. We have an involution τ on vector bundles over X , given by $\tau E = u^*\bar{E}$. The group $KR(X)$ is then the Grothendieck group of isomorphism classes of objects in the skewed category. A $\mathbb{Z}/2$ -spectrum KR representing this cohomology theory is constructed in [Dug05]. This construction begins by noting that one can put a $\mathbb{Z}/2$ -action on $\mathbb{Z} \times BU$ using the conjugation action on U . One can then use this action to give a $\mathbb{Z}/2$ -spectrum in a similar way to Corollary 8.1.5, where we give $SO(2)$ -spectra with extra information the structure of $O(2)$ -spectra. We are simply noting that these constructions are similar to those in this thesis; it could be interesting to study the relation between the skewed categories of vector bundles and KR in greater detail.

Remark 7.4.5 There is an adjunction $\text{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \text{Fun}(\mathcal{C}, \text{Fun}(\mathcal{D}, \mathcal{E}))$ for (small) categories \mathcal{C} , \mathcal{D} , \mathcal{E} . Applying this in the case of the group categories G and H (these each have one object and morphisms correspond to group elements), we see that the category of $G \times H$ -objects in \mathcal{C} , $\text{Fun}(G \times H, \mathcal{C})$, is isomorphic to the category of G -objects in the category of H -objects of \mathcal{C} $\text{Fun}(G, \text{Fun}(H, \mathcal{C}))$. Now consider a semi-direct product $G \ltimes H$ (so G acts on H). The skewed category (or rather, its generalisation to general G , see Remark 7.3.17) provides us with an equivalence between $G \# \text{Fun}(H, \mathcal{C})$ and $\text{Fun}(G \ltimes H, \mathcal{C})$. We can also link this example back to skew group rings, by [MR01, 1.5.7] there is an isomorphism of rings $R(G \ltimes H) \cong RH \# G$. Thus $R(G \ltimes H)\text{-mod}$ is equivalent to $(RH \# G)\text{-mod}$ and by the example above on skew group rings this is equivalent to $G \# (RH\text{-mod})$.

Chapter 8

Cyclic $O(2)$ -Spectra

We study cyclic $O(2)$ -spectra and relate this category to rational $SO(2)$ -spectra. The first section considers $SO(2)$ -spectra in general and proves that this category has an involution (Lemma 8.1.7). The second section applies this work to construct an involution on $\iota^*S_{\mathbb{Q}}\text{-mod}$, a particular model for rational $SO(2)$ -spectra. We have chosen this model as every object is fibrant, which is necessary for Theorem 9.1.2. The highlight of this chapter is Theorem 8.2.5 which describes cyclic $O(2)$ -spectra in terms of the skewed category of $\iota^*S_{\mathbb{Q}}\text{-mod}$.

8.1 An Involution on $SO(2)$ -Spectra

We want to understand cyclic $O(2)$ -spectra by proving an analogue of Example 7.4.2. This section makes substantial progress in that direction with Theorem 8.1.4 and Corollary 8.1.5. We will use these in the next section to prove this chapter's main result: Theorem 8.2.5. We have to work harder than our space level example to relate $SO(2)$ -spectra to $O(2)$ -spectra since G -equivariant spectra aren't simply G -objects in the category of non-equivariant spectra.

We now need to specify the universe, so we write $GM(U)$ for G -equivariant S -modules indexed on the G -universe U . Fix U to be a complete $O(2)$ -universe and write ι^* for $\iota_{SO(2)}^*$, this is a functor $O(2)\mathcal{M}(U) \rightarrow SO(2)\mathcal{M}(\iota^*U)$, note that ι^*U is a complete $SO(2)$ -universe. We have the change of groups functor $j^*: SO(2)\mathcal{M}(\iota^*U) \rightarrow SO(2)\mathcal{M}(j^*\iota^*U)$ induced from j ($j^*\iota^*U$ is ι^*U with the opposite $SO(2)$ action). Since an involution must have the domain equal to the codomain, we cannot just use j^* , we need to change back to the universe ι^*U . For G -universes U and U' , we have a space $\mathcal{J}(U, U')$ of linear isometries $U \rightarrow U'$, with G acting by conjugation. An object of $GM(U)$ is a G -May spectrum with an action of $\mathcal{J}(U, U) \ltimes (-)$ that is also an S -module.

Definition 8.1.1 For universes U and U' we define $I_U^{U'} = \mathcal{J}(U, U') \ltimes_{\mathcal{J}(U, U)} (-)$ (see [EM97, Definition 2.1] or [May96, Chapter XXIV, Definition 3.4]).

There is a strong symmetric monoidal pair ([EM97, Theroem 1.1] or [May96, Chapter XXIV, Theorem 3.7])

$$I_U^{U'} : GM(U) \rightleftarrows GM(U') : I_{U'}^U.$$

So our first candidate for an involution is $I_{j^* \iota^* U}^{\iota^* U} j^*$. But this is not good enough for our work, so we introduce a more classical version of change of universe that is easy to work with. Since we have used an asterisk to denote change of groups functors we use a \dagger to indicate change of universe functors. The following is [LMSM86, Chapter I, Proposition 2.5].

Definition 8.1.2 Let $f : U \rightarrow U'$ be a G -linear isometry between universes. For a G -spectrum X indexed on U' we define $f^\dagger X$ a G -spectrum indexed on U by $(f^\dagger X)(V) = X(f(V))$, for V an indexing space of U . For Y , a G -spectrum indexed on U , we let $(f_\dagger Y)(W) = Y(f^{-1}(W))$ (W an indexing space of U') and we have $f_\dagger Y$ indexed on U' . There is an adjoint pair

$$f_\dagger : GM(U) \rightleftarrows GM(U') : f^\dagger$$

.

The following result relates these two definitions, in our case it tells us that we can replace the complicated functor $I_{j^* \iota^* U}^{\iota^* U}$ with the more intuitive change of universe functors f_\dagger and f^\dagger . We also note [EKMM97, Appendix A, Proposition 5.3] which states that $f \ltimes X = f_\dagger X$ (considering f as a point in $\mathcal{S}(U, U')$).

Lemma 8.1.3 Let $f : U \rightarrow U'$ be an isomorphism of G -universes. Then there are natural isomorphisms $f_\dagger E \cong I_U^{U'} E$ and $f^\dagger E' \cong I_{U'}^U E'$ for $E \in G\mathcal{S}U[\mathbb{L}]$ and $E' \in G\mathcal{S}U'[\mathbb{L}']$.

Proof This result is taken from [May98, Lemma 3.5]. ■

Any reflection $h \in O(2)$ gives an $SO(2)$ -equivariant isomorphism of universes $h : \iota^* U \rightarrow j^* \iota^* U$. As h changes one obtains different isomorphisms, but since any two reflections are conjugate by a rotation one can see how these maps will be related. Thus we have a change of universe functor $h^\dagger : SO(2)\mathcal{M}(j^* \iota^* U) \rightarrow SO(2)\mathcal{M}(\iota^* U)$ that is isomorphic to $I_{\iota^* U}^{j^* \iota^* U}$. From here onwards we let h be some fixed reflection in $O(2)$, Theorem 8.2.5 will show that this choice is not important.

Theorem 8.1.4 For an $O(2)$ -spectrum X indexed on an $O(2)$ -universe U , we have an $SO(2)$ -map (natural in $O(2)$ -maps of X) $h : \iota^* X \rightarrow h^\dagger j^* \iota^* X$. The composite $(h^\dagger j^* h) \circ h$ is the identity map of X .

Proof We begin with the definition of the map on the levels of $\iota^* X$, then show this is a map of spectra, a map of $\iota^* \mathbb{L}$ -spectra and of $SO(2)$ -equivariant S -modules in turn. Take an indexing space $\iota^* V$ of $\iota^* U$, then we have a map

$$h : (\iota^* X)(\iota^* V) = \iota^*(X(V)) \longrightarrow j^* \iota^*(X(V)) = (h^\dagger j^* \iota^* X)(\iota^* V).$$

The last equality is simple: V is a sub-inner- $O(2)$ -product space of U , hence $hV = V$ as sets, but as usual the action of $SO(2)$ is inverted, so $hV = j^* V$ as $SO(2)$ -inner

product spaces. Now we show that this gives a map of spectra: take $\iota^*W \subset \iota^*V$ and recall that the structure maps for $h^\dagger j^* \iota^* X$ are given by $j^* \iota^* \sigma \circ h \wedge \text{Id}$ as shown below.

$$\begin{aligned} S^{\iota^*W - \iota^*V} \wedge (h^\dagger j^* \iota^* X)(\iota^*V) &\xrightarrow{h \wedge \text{Id}} S^{j^* \iota^*W - j^* \iota^*V} \wedge (h^\dagger j^* \iota^* X)(\iota^*V) \\ &= j^* \iota^* (S^{W-V} \wedge X(V)) \\ &\xrightarrow{j^* \iota^* \sigma} j^* \iota^* (X(W)) \end{aligned}$$

The relevant equation to check that h is a map of spectra is $j^* \iota^* \sigma \circ h \wedge \text{Id} \circ \text{Id} \wedge h = h \iota^* \sigma$. As maps of the underlying sets, $j^* \iota^* \sigma = \iota^* \sigma = \sigma$, so the above equation is precisely the requirement that the structure maps σ of X are $O(2)$ -equivariant.

Now we show that h is a map of $\iota^* \mathbb{L}$ -spectra. We begin with a diagram which may help to explain the following work. We use ρ to denote the action of $\mathcal{S}(U, U)$ on X . The idea behind a half-twisted smash product is that for each $\theta \in \mathcal{S}(\iota^*U, \iota^*U)$ we have a map of non-equivariant spectra $\rho(\theta): X \rightarrow \theta^\dagger X$. The diagram below commutes because ρ is an $O(2)$ -equivariant map, noting that $h^\dagger \theta^\dagger = (\theta h)^\dagger$ which is equal to $(h h \theta h)^\dagger = (h \theta h)^\dagger h^\dagger$.

$$\begin{array}{ccc} X & \xrightarrow{\rho(\theta)} & \theta^\dagger X \\ \downarrow h & & \downarrow h \\ h^\dagger X & \xrightarrow{\rho(h \theta h)} & (h \theta h)^\dagger X \end{array}$$

From ρ we have $\iota^* \rho$, the action of $\mathcal{S}(\iota^*U, \iota^*U)$ on ι^*X . We now define ρ' , the action of $\mathcal{S}(\iota^*U, \iota^*U)$ on $h^\dagger j^* \iota^* X$. We have an isomorphism of functors

$$(\mathcal{S}(\iota^*U, \iota^*U) \times \{h^{-1}\}) \ltimes (-) \cong \mathcal{S}(\iota^*U, \iota^*U) \ltimes h^\dagger(-).$$

We can apply conjugation by h (more properly by h^{-1}) to obtain a homeomorphism of $SO(2)$ -spaces:

$$C_h: \mathcal{S}(\iota^*U, \iota^*U) \times \{h^{-1}\} \rightarrow \{h^{-1}\} \times \mathcal{S}(j^* \iota^*U, j^* \iota^*U).$$

We combine these to give the structure map ρ' as follows, using $j^* \iota^* \rho$, the action of $\mathcal{S}(j^* \iota^*U, j^* \iota^*U)$ on $j^* \iota^* X$.

$$\mathcal{S}(\iota^*U, \iota^*U) \ltimes h^\dagger j^* \iota^* X \rightarrow (\{h^{-1}\} \times \mathcal{S}(j^* \iota^*U, j^* \iota^*U)) \ltimes j^* \iota^* X \rightarrow h^\dagger j^* \iota^* X$$

The following diagram commutes since $\rho: \mathcal{S}(U, U) \ltimes X \rightarrow X$ is an equivariant map (note that h acts by conjugation on $\mathcal{S}(U, U)$).

$$\begin{array}{ccc} \mathcal{S}(\iota^*U, \iota^*U) \ltimes \iota^* X & \xrightarrow{\iota^* \rho} & \iota^* X \\ \downarrow \text{Id} \wedge h & & \downarrow h \\ \mathcal{S}(\iota^*U, \iota^*U) \ltimes h^\dagger j^* \iota^* X & \xrightarrow{\rho'} & h^\dagger j^* \iota^* X \end{array}$$

Thankfully we can now turn to easier considerations and show that h is a map of $SO(2)$ -equivariant S -modules. Since the functor $h^\dagger j^*$ is isomorphic to the strong

monoidal functor $I_{j^* \iota^* U}^{*U} j^*$, $h^\dagger j^* \iota^* X$ is a module over $h^\dagger j^* \iota^* S$. We have a map of ring spectra $h: \iota^* S \rightarrow h^\dagger j^* \iota^* S$ and we can use this map to give $h^\dagger j^* \iota^* X$ an $\iota^* S$ -module structure. With this in place we have to check that the following diagram commutes.

$$\begin{array}{ccc} \iota^* S \wedge \iota^* X & \xrightarrow{\iota^* \nu} & \iota^* X \\ \downarrow \text{Id} \wedge h & & \downarrow h \\ \iota^* S \wedge h^\dagger j^* \iota^* X & \xrightarrow{\nu'} & h^\dagger j^* \iota^* X \end{array}$$

Where $\nu' = \iota^* \nu (h \wedge \text{Id})$ is the $\iota^* S$ -action map on $h^\dagger j^* \iota^* X$. It is clear that this diagram commutes precisely when ν is $O(2)$ -equivariant. Thus $h: \iota^* X \rightarrow h^\dagger j^* \iota^* X$ is a map of $SO(2)$ -spectra, we now prove it is a map of order two. Consider the composition

$$(h^\dagger j^* h) \circ h: \iota^*(X(V)) \rightarrow j^* \iota^*(X(V)) \rightarrow \iota^*(X(V))$$

and we note that (on the level of sets) this is simply a double application of the automorphism h to (the $O(2)$ -space) $X(V)$, hence the composite $(h^\dagger j^* h) \circ h$ is the identity map of X . ■

Corollary 8.1.5 *An $SO(2)$ spectrum Y with a map $f: Y \rightarrow h^\dagger j^* Y$ such that $(h^\dagger j^* f) \circ f = \text{Id}_Y$ can be given the structure of an $O(2)$ spectrum. This construction depends naturally on Y and f .*

Proof This is contained in the proof of the above theorem, see the proof of Theorem 8.2.5 for some details. ■

Lemma 8.1.6 *The functor $h^\dagger j^*$ preserves weak equivalences, cofibrations and fibrations of GM .*

Proof All we need to show is that $h^\dagger j^*$ takes generating (acyclic) cofibrations to (acyclic) cofibrations since the right adjoint of $h^\dagger j^*$ is $h^\dagger j^*$ itself. We let E be a generalised sphere spectrum for $SO(2)$. Inversion gives an isomorphism $SO(2)/H_+ \rightarrow j^* SO(2)/H_+$ and hence we have an isomorphism $E \rightarrow h^\dagger j^* E$. This gives an isomorphism of maps between the generating cofibration $E \rightarrow CE$ and its image $h^\dagger j^* E \rightarrow h^\dagger j^* CE$. Thus the map $h^\dagger j^* E \rightarrow h^\dagger j^* CE$ is a cofibration. We have proven that $h^\dagger j^*$ preserves cofibrations, the same argument suffices to show that it also preserves acyclic cofibrations. ■

Lemma 8.1.7 *The functor $h^\dagger j^*$ is an involution on $SO(2)\mathcal{M}$, hence the category $(SO(2)\mathcal{M}, h^\dagger j^*)$ is an involutory model category.*

Proof The functor $h^\dagger j^*$ preserves the underlying sets of $SO(2)$ -spectra, hence $h^\dagger j^* \circ h^\dagger j^* = \text{Id}$ and we have an involution on $SO(2)\mathcal{M}$. Lemma 8.1.6 proves that $h^\dagger j^*$ is a left Quillen functor, so we have a model category with involution. ■

8.2 The category $\tau\#\iota^*S_{\mathbb{Q}}\text{-mod}$

We prove that the category of cyclic spectra is Quillen equivalent to the skewed category of rational $SO(2)$ -spectra (Theorem 8.2.5). Our model for cyclic spectra will be modules over $S_{\mathbb{Q}} \in O(2)\mathcal{M}(U)$ (for U a complete $O(2)$ -universe) with model structure lifted from the \mathcal{C} -model structure on $O(2)\mathcal{M}(U)$. This is written as $S_{\mathbb{Q}}\text{-mod}(\mathcal{C})$ and is Quillen equivalent to $\mathcal{C}\mathcal{M}_{\mathbb{Q}}$ by Proposition 6.1.6. Our model for rational $SO(2)$ -spectra will be modules over $\iota^*S_{\mathbb{Q}}\text{-mod}$ in $SO(2)\mathcal{M}(\iota^*U)$. There is a ring map of order two $\alpha: \iota^*S_{\mathbb{Q}} \rightarrow h^\dagger j^* \iota^*S_{\mathbb{Q}}$, thus we can consider the functor $\tau = \alpha^* h^\dagger j^*$, this will be our involution on $\iota^*S_{\mathbb{Q}}\text{-mod}$. This section will prove the claim that cyclic $O(2)$ -spectra are rational $SO(2)$ -spectra with extra structure. In detail, this structure is a map of order two $X \rightarrow \tau X$, so $\tau\#\iota^*S_{\mathbb{Q}}\text{-mod}$ will be Quillen equivalent to $S_{\mathbb{Q}}\text{-mod}(\mathcal{C})$. The proof of this statement is quite long, so we break down the construction of $\tau\#\iota^*S_{\mathbb{Q}}\text{-mod}$ into several results.

We have made this choice of categories so that every object is fibrant, thus meeting one technical condition of Theorem 9.1.2. It is no more difficult to prove Theorem 8.2.5 with this choice of categories than with any other. The difficult part of this section is proving that $h^\dagger j^*$ is a monoidal involution. We would want this result in any case, to know that our description of cyclic spectra as a skewed category is a *monoidal* Quillen equivalence.

Lemma 8.2.1 *The identity map is a natural transformation $h^\dagger j^* \rightarrow j^* h^\dagger$.*

Proof We draw the following diagram which obviously commutes.

$$\begin{array}{ccc} SO(2)\mathcal{M}(\iota^*U) & \xrightarrow{h^\dagger} & SO(2)\mathcal{M}(j^*\iota^*U) \\ \downarrow j^* & & \downarrow j^* \\ SO(2)\mathcal{M}(j^*\iota^*U) & \xrightarrow{h^\dagger} & SO(2)\mathcal{M}(\iota^*U) \end{array}$$

■

In the following we will need the category $G\mathcal{J}(U'; U)$, as defined in [EKMM97, Appendix A, Section 2], for G -universes U and U' . An object \mathcal{E} of this category is a collection of spectra $\mathcal{E}_V \in G\mathcal{M}(U')$ where V runs over indexing spaces of U , with a transitive system of isomorphisms $\Sigma^{W-V}\mathcal{E}_W \rightarrow \mathcal{E}_V$. Morphisms are then just a family of morphisms in $G\mathcal{M}(U')$ compatible with the structure maps. We let $X \bar{\wedge} Y$ denote the external smash product of a pair of spectra X and Y in $G\mathcal{M}(U)$, it is a spectrum indexed on a universe $U \oplus U$ defined by $X \bar{\wedge} Y(V \oplus W) = X(V) \wedge Y(W)$.

Lemma 8.2.2 *The functor $h^\dagger j^*$ is a strong monoidal functor.*

Proof Since h^\dagger is naturally isomorphic to $I_{j^*\iota^*U}^{*\iota^*U}$, we know that $h^\dagger j^*$ is strong monoidal. We give a direct proof of this, as we will later need to show that $h^\dagger j^*$ is a monoidal involution. We require a natural transformation $m: h^\dagger j^*(-) \wedge h^\dagger j^*(-) \rightarrow h^\dagger j^*(- \wedge -)$. Take $SO(2)$ -spectra X and Y , we compare $h^\dagger j^*X \wedge h^\dagger j^*Y$ to $h^\dagger j^*(X \wedge Y)$. The object

$h^\dagger j^* X \wedge h^\dagger j^* Y$ is defined to be the first term below and we have an isomorphism

$$\begin{aligned} \mathcal{J}(\iota^* U \oplus \iota^* U, \iota^* U) &\ltimes_{\mathcal{J}(\iota^* U, \iota^* U)^2} (h^\dagger j^* X \bar{\wedge} h^\dagger j^* Y) \\ &\downarrow \cong \\ \mathcal{J}(\iota^* U \oplus \iota^* U, \iota^* U) &\ltimes_{(\mathcal{J}(\iota^* U, \iota^* U) \times \{h\})^2} (j^* X \bar{\wedge} j^* Y). \end{aligned}$$

We then apply conjugation by h to obtain an isomorphism from the above to

$$\mathcal{J}(\iota^* U \oplus \iota^* U, \iota^* U) \ltimes_{(\{h\} \times \mathcal{J}(j^* \iota^* U, j^* \iota^* U))^2} (j^* X \bar{\wedge} j^* Y).$$

Because this is a coequaliser this is isomorphic to

$$(\mathcal{J}(\iota^* U \oplus \iota^* U, \iota^* U) \times \{h \oplus h\}) \ltimes_{\mathcal{J}(j^* \iota^* U, j^* \iota^* U)^2} (j^* X \bar{\wedge} j^* Y).$$

Another application of conjugation by h gives the first term below and then we have the isomorphism

$$\begin{aligned} (\{h\} \times \mathcal{J}(j^* \iota^* U \oplus j^* \iota^* U, j^* \iota^* U)) &\ltimes_{\mathcal{J}(j^* \iota^* U, j^* \iota^* U)^2} (j^* X \bar{\wedge} j^* Y) \\ &\downarrow \cong \\ h^\dagger \mathcal{J}(j^* \iota^* U \oplus j^* \iota^* U, j^* \iota^* U) &\ltimes_{\mathcal{J}(j^* \iota^* U, j^* \iota^* U)^2} (j^* X \bar{\wedge} j^* Y). \end{aligned}$$

We combine these maps to obtain $m_1: h^\dagger j^* X \wedge h^\dagger j^* Y \rightarrow h^\dagger (j^* X \wedge j^* Y)$. We know that the change of groups functor is strong monoidal hence there is a natural transformation $m_2: h^\dagger (j^* X \wedge j^* Y) \rightarrow h^\dagger j^* (X \wedge Y)$. The composite of m_2 and m_1 gives m . \blacksquare

Proposition 8.2.3 *The category $(SO(2)\mathcal{M}, h^\dagger j^*)$ is an involutory monoidal model category.*

Proof We have already shown that $(SO(2)\mathcal{M}, h^\dagger j^*)$ is an involutory model category, and $h^\dagger j^*$ is a strong monoidal functor. Thus, all that remains is to prove is the involutory condition $h^\dagger j^* m \circ m(h^\dagger j^* \wedge h^\dagger j^*) = \text{Id}$ and to specify a map of order two $S_{SO(2)} \rightarrow h^\dagger j^* S_{SO(2)}$. This second condition is simple: consider the $O(2)$ -equivariant sphere spectrum $S_{O(2)}$. The unit of $SO(2)\mathcal{M}$ is $\iota^* S_{O(2)} = S_{SO(2)}$. Hence, by Theorem 8.1.4, we have a ring map of order two $\alpha: \iota^* S_{O(2)} \rightarrow h^\dagger j^* \iota^* S_{O(2)}$.

What remains is a technical proof, one that requires us to look in great detail at the smash product of $SO(2)\mathcal{M}$. We must check the equation: $h^\dagger j^* m \circ m(h^\dagger j^* \wedge h^\dagger j^*) = \text{Id}$. Our method of proof is as follows. Since $m = h^\dagger m_2 \circ m_1$, we prove a similar condition for each of the factors in turn. Let X and Y be in $SO(2)\mathcal{M}(\iota^* U)$, then by the proof that h^\dagger is a monoidal functor we see that

$$\text{Id}_{X \wedge Y} = h^\dagger m_1 \circ m_1(h^\dagger \wedge h^\dagger): h^\dagger h^\dagger X \wedge h^\dagger h^\dagger Y \longrightarrow h^\dagger h^\dagger (X \wedge Y).$$

We will prove below a similar condition on m_2 :

$$\text{Id}_{X \wedge Y} = j^* m_2 \circ m_2(j^* \wedge j^*): j^* j^* X \wedge j^* j^* Y \longrightarrow j^* j^* (X \wedge Y).$$

Then assuming that m_1 and m_2 commute, we have the following commutative diagram.

$$\begin{array}{ccc}
h^\dagger j^* h^\dagger j^* X \wedge h^\dagger j^* h^\dagger j^* Y & \xrightarrow{=} & h^\dagger h^\dagger j^* j^* X \wedge h^\dagger h^\dagger j^* j^* Y \\
m_1 \downarrow & & m_1 \downarrow \\
h^\dagger(j^* h^\dagger j^* X \wedge j^* h^\dagger j^* Y) & \xrightarrow{=} & h^\dagger(h^\dagger j^* j^* X \wedge h^\dagger j^* j^* Y) \\
h^\dagger m_2 \downarrow & & h^\dagger m_1 \downarrow \\
h^\dagger j^*(h^\dagger j^* X \wedge h^\dagger j^* Y) & & h^\dagger h^\dagger(j^* j^* X \wedge j^* j^* Y) \\
h^\dagger j^* m_1 \downarrow & & h^\dagger h^\dagger m_2 \downarrow \\
h^\dagger j^* h^\dagger(j^* X \wedge j^* Y) & \xrightarrow{=} & h^\dagger h^\dagger j^*(j^* X \wedge j^* Y) \\
h^\dagger j^* h^\dagger m_2 \downarrow & & h^\dagger h^\dagger j^* m_2 \downarrow \\
h^\dagger j^* h^\dagger j^*(X \wedge Y) & \xrightarrow{=} & h^\dagger h^\dagger j^* j^*(X \wedge Y)
\end{array}$$

The left hand vertical composite is $h^\dagger j^* m \circ m(h^\dagger j^* \wedge h^\dagger j^*)$, so once we have shown that m_1 and m_2 commute and that m_2 satisfies the above condition, we will have our result.

We need a more explicit description of m_2 , so we write the half-twisted smash product in terms of a colimit over pairs of indexing spaces V and W of the spectra $\mathcal{E}_{V \oplus W} \wedge X(V) \wedge Y(W)$ for a particular $\mathcal{E} \in SO(2)\mathcal{J}(\iota^* U \oplus \iota^* U; \iota^* U)$. That is,

$$X \wedge Y = \operatorname{Colim}_{V \oplus W} \left(\mathcal{E}_{V \oplus W} \wedge X(V) \wedge Y(W) \right)$$

Now we can describe the action of m_2 in more detail, consider

$$j^* X \wedge j^* Y = \operatorname{Colim}_{V \oplus W} \left(j^*(\mathcal{E}_{V \oplus W}) \wedge j^* X(V) \wedge j^* Y(W) \right)$$

m_2 acts on this colimit termwise, using the obvious isomorphism:

$$n: j^*(\mathcal{E}_{V \oplus W}) \wedge j^* X(V) \wedge j^* Y(W) \cong j^* \left((\mathcal{E}_{V \oplus W}) \wedge X(V) \wedge Y(W) \right).$$

Applying this isomorphism twice we obtain an isomorphism

$$n^2: j^* j^*(\mathcal{E}_{V \oplus W}) \wedge j^* j^* X(V) \wedge j^* j^* Y(W) \cong j^* j^* \left((\mathcal{E}_{V \oplus W}) \wedge X(V) \wedge Y(W) \right).$$

and this map is the identity. Hence the above condition for m_2 holds. Now we note that the natural transformations m_1 and m_2 commute, in the description of the smash product above, the map m_1 only acts on the terms $\mathcal{E}_{V \oplus W}$. Hence by the naturality properties of a colimit, the claim follows. Thus, since m is the composite of the commuting maps m_1 and m_2 it satisfies the involutory condition as desired. \blacksquare

We define $\tau = \alpha^* h^\dagger j^*$, an involution on $\iota^* S_{\mathbb{Q}}\text{-mod}$, using the ring map given in the above proof $\alpha: \iota^* S_{\mathbb{Q}} \rightarrow h^\dagger j^* \iota^* S_{\mathbb{Q}}$.

Proposition 8.2.4 *The category of rational $SO(2)$ -spectra, $\iota^* S_{\mathbb{Q}}\text{-mod}$, is an involutory symmetric monoidal model category with involution τ .*

Proof This follows from Proposition 7.3.16 since $(SO(2)\mathcal{M}, h^\dagger j^*)$ is an involutory monoidal model category. \blacksquare

Thus, we have a model category $\tau \# \iota^* S_{\mathbb{Q}}\text{-mod}$. Using notation introduced below we can also write $I(S_{\mathbb{Q}})\text{-mod}$ for $\tau \# \iota^* S_{\mathbb{Q}}\text{-mod}$. Recall that h is our fixed reflection in $O(2)$, j^* is the change of groups functor (from the map $t \rightarrow t^{-1}$ of $SO(2)$), h^\dagger the change of universe functor induced by h and $\alpha: \iota^* S_{\mathbb{Q}} \rightarrow h^\dagger j^* \iota^* S_{\mathbb{Q}}$ is a ring map. We use the category $S_{\mathbb{Q}}\text{-mod}(\mathcal{C})$, of $S_{\mathbb{Q}}$ -modules in $O(2)\mathcal{M}$, with weak equivalences those maps which are π_*^H -isomorphisms for $H \leq SO(2)$.

Theorem 8.2.5 *There is a strong symmetric monoidal Quillen equivalence*

$$C : \tau \# \iota^* S_{\mathbb{Q}}\text{-mod} \rightleftarrows S_{\mathbb{Q}}\text{-mod}(\mathcal{C}) : I$$

where $IX = h: \iota^* X \rightarrow \alpha^* h^\dagger j^* \iota^* X$, $If = \iota^* f$ and C is defined in the proof.

Proof Almost all of the work has been done in Theorem 8.1.4 and Corollary 8.1.5. We are merely going to repeat the arguments to construct a map of $O(2)$ -spectra from a map in the skewed category and then check that we have a well-behaved adjunction. But first, let us recap the construction of an $O(2)$ -spectrum from an object of the skewed category. Take $h: X \rightarrow \alpha^* h^\dagger j^* X$ (X an $SO(2)$ -spectrum and h a map of order two) then $CX(V) = (X \circ \iota^*)(V) = X(\iota^* V)$ as a topological space. We give $CX(V)$ an $O(2)$ -action by letting $h \in O(2)$ our chosen reflection act as $h: X(\iota^* V) \rightarrow j^*(X(\iota^* V))$. Now we must add the structure of a prespectrum: $\sigma: S^{W-V} \wedge CX(V) \rightarrow CX(W)$ is defined by the corresponding structure map of X , that this is an $O(2)$ -map is encoded in the fact that h is a map of prespectra. It is clear that this gives a spectrum (that is, the adjoints of the structure maps are equivariant homeomorphisms).

We define the \mathbb{L} -spectrum action using the underlying set of the structure map for X and then check that this gives an $O(2)$ -map. Repeating this process twice more we see that CX is an $O(2)$ -equivariant S -module and an $S_{\mathbb{Q}}$ -module. Now for the action of C on maps, with the above construction we can simply define $Cf_V = f_V$ (as a set map) for f a map as below.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h_X & & \downarrow h_Y \\ h^\dagger j^* X & \xrightarrow{h^\dagger j^* f} & h^\dagger j^* Y \end{array}$$

It is routine to check that Cf defines an $O(2)$ -equivariant map of $S_{\mathbb{Q}}$ -modules. That these are adjoint functors is immediate, the pair (C, I) is actually an equivalence of categories. Now we consider the model structures, for which we need the diagram below (left adjoints will be on the top and left). Recall that $O(2)\mathcal{M}(SO(2))$ is the category

$O(2)\mathcal{M}$ with the \mathcal{C} -model structure as defined in Theorem 3.4.4. It should be clear that the right adjoints of the square and triangle commute, hence so do the left adjoints (up to natural isomorphism). The pairs $(S_{\mathbb{Q}} \wedge (-), U_1)$ and $(\iota^* S_{\mathbb{Q}} \wedge (-), U_2)$ are the free module and underlying spectrum pairs.

$$\begin{array}{ccc}
 & \tau \# \iota^* S_{\mathbb{Q}}\text{-mod} & \\
 \begin{array}{c} \nearrow C \\ \searrow I \end{array} & & \begin{array}{c} \nwarrow \mathbb{D} \\ \swarrow \mathbb{P} \end{array} \\
 S_{\mathbb{Q}}\text{-mod}(\mathcal{C}) & \xrightleftharpoons[\iota^*]{0(2)_+ \wedge_{SO(2)}(-)} & \iota^* S_{\mathbb{Q}}\text{-mod} \\
 \begin{array}{c} \uparrow S_{\mathbb{Q}} \wedge (-) \\ \downarrow U_1 \end{array} & & \begin{array}{c} \downarrow \iota^* S_{\mathbb{Q}} \wedge (-) \\ \uparrow U_2 \end{array} \\
 O(2)\mathcal{M}(SO(2)) & \xrightleftharpoons[\iota^*]{0(2)_+ \wedge_{SO(2)}(-)} & SO(2)\mathcal{M}
 \end{array}$$

We show that I is a right Quillen functor. Take a fibration or acyclic fibration f in $S_{\mathbb{Q}}\text{-mod}(\mathcal{C})$, then If is a fibration or acyclic fibration exactly when $\mathbb{P}If = \iota^* f$ in $\iota^* S_{\mathbb{Q}}\text{-mod}$ is. Now $\iota^* f$ is a fibration or acyclic fibration in $\iota^* S_{\mathbb{Q}}\text{-mod}$ precisely when it is so in $SO(2)\mathcal{M}$. Since $U_1: S_{\mathbb{Q}}\text{-mod}(\mathcal{C}) \rightarrow O(2)\mathcal{M}(SO(2))$ and $\iota^*: O(2)\mathcal{M}(SO(2)) \rightarrow SO(2)\mathcal{M}$ are a right Quillen functors, this result follows. Now we must show that (C, I) is a Quillen equivalence, the above argument shows that I detects and preserves all weak equivalences (since U_1 and $\iota^*: O(2)\mathcal{M}(SO(2)) \rightarrow SO(2)\mathcal{M}$ do so). In fact, C also preserves all weak equivalences, since $\iota^* \circ C = \mathbb{P}$. Now we show that the composite $X \rightarrow ICX \rightarrow I\widehat{f}CX$ is a weak equivalence in $\tau \# \iota^* S_{\mathbb{Q}}\text{-mod}$, the first arrow is an isomorphism, the second is I applied to a weak equivalence, hence a weak equivalence.

We prove that the left adjoint C is strong symmetric monoidal. Let (X, u) and (Y, v) be objects of the skewed category. The underlying $SO(2)$ -spectra of $C(X, u) \wedge C(Y, v)$ and $C((X, u) \wedge (Y, v))$ are clearly isomorphic and given by $\mathcal{S}(U \oplus U, U) \ltimes_{\mathcal{S}(U, U)^2} X \bar{\wedge} Y$. The action of $O(2)$ in each case is given by conjugation on $\mathcal{S}(U \oplus U, U)$, by u on X and by v on Y . Thus we see that these objects are isomorphic as $O(2)$ -spectra. The symmetry statement is clear and $C(\iota^* S_{\mathbb{Q}}, \alpha) \cong S_{\mathbb{Q}}$ so C is strong symmetric monoidal. It remains to check that if $(\widehat{c}\iota^* S_{\mathbb{Q}}, \widehat{c}\alpha)$ is a cofibrant replacement of the unit, then $C(\widehat{c}\iota^* S_{\mathbb{Q}}, \widehat{c}\alpha) \rightarrow C(\iota^* S_{\mathbb{Q}}, \alpha)$ is a weak equivalence. This holds since C preserves all weak equivalences. \blacksquare

Theorem 8.2.6 *The categories \mathcal{M} , $SO(2)\mathcal{M}$, $O(2)\mathcal{M}$, $\iota^* S_{\mathbb{Q}}\text{-mod}$ and $S_{\mathbb{Q}}\text{-mod}$ are all closed symmetric Sp_+^{Σ} -algebras.*

Proof Each of these categories is a closed symmetric monoidal category, hence they are algebras over themselves, we can now use the following series of adjoints to pull this structure back to Sp_+^{Σ} , symmetric spectra of simplicial sets with the positive model structure. The geometric realisation and singular complex adjunction $(|-|, \text{Sing})$ is a strong symmetric monoidal Quillen equivalence between Sp_+^{Σ} and $Sp_+^{\Sigma}(\mathcal{T}_*)$ ([SS03a, Figure 7.1]). We have the strong symmetric monoidal Quillen equivalence

$$\mathbb{P}: Sp_+^{\Sigma}(\mathcal{T}_*) \rightleftarrows \mathcal{S}\mathcal{S}_+: \mathbb{U}$$

from [MMSS01]. Combining these gives $\mathcal{I}\mathcal{S}_+$ the structure of a closed symmetric Sp_+^Σ -algebra. Thus we can use the adjunction $(\mathbb{N}, \mathbb{N}^\#)$ to see that \mathcal{M} is a closed symmetric Sp_+^Σ -algebra.

When working G -equivariantly we use the composite functor $i_*\varepsilon_G^*: \mathcal{I}\mathcal{S}_+ \rightarrow G\mathcal{I}\mathcal{S}_+$ as defined in [MM02, Chapter V, Proposition 3.4], which states that this functor is part of a Quillen pair $(i_*\varepsilon_G^*, (i^*(-))^G)$. This is a strong symmetric monoidal adjunction, as noted on [MM02, Chapter V, Page 80]. Finally, for $S_{\mathbb{Q}}\text{-mod}$ and $\iota^*S_{\mathbb{Q}}\text{-mod}$ we use the free module functor as well. We illustrate for $S_{\mathbb{Q}}\text{-mod}$ below.

$$Sp_+^\Sigma \xrightleftharpoons[\text{Sing} \circ \mathbb{U}]{\mathbb{P} \circ |-|} \mathcal{I}\mathcal{S}_+ \xrightleftharpoons[(i^*(-))^{O(2)}]{i_*\varepsilon_{O(2)}^*} O(2)\mathcal{I}\mathcal{S}_+ \xrightleftharpoons[\mathbb{N}^\#]{\mathbb{N}} O(2)\mathcal{M} \xrightleftharpoons[U]{S_{\mathbb{Q}} \wedge (-)} S_{\mathbb{Q}}\text{-mod}$$

For a pair of $S_{\mathbb{Q}}$ -modules X and Y , the Sp_+^Σ -function object $\underline{\text{Hom}}(X, Y) \in Sp_+^\Sigma$ is given by $\text{Sing } \mathbb{U}(i^*\mathbb{N}^\#UF_{S_{\mathbb{Q}}}(X, Y))^{O(2)}$. ■

Lemma 8.2.7 *The Quillen equivalence (C, I) is an adjunction of closed symmetric $\text{Id} \# Sp_+^\Sigma$ -algebras.*

Proof The composite functor $i_{SO(2)}: Sp_+^\Sigma \rightarrow \iota^*S_{\mathbb{Q}}\text{-mod}$, as defined above, is an involutory functor. This can be seen most easily by noting that $\iota_{SO(2)}^* \circ i_{O(2)} \cong i_{SO(2)}$ (where $i_{O(2)}$ is the functor Sp_+^Σ to $S_{\mathbb{Q}}\text{-mod}$ as constructed above) and applying Theorem 8.1.4, see Lemma 9.1.3. It follows that $\tau \# \iota^*S_{\mathbb{Q}}\text{-mod}$ is a closed symmetric $\text{Id} \# Sp_+^\Sigma$ -algebra. The result then follows since $O(2)\mathcal{M}(SO(2))$ is a closed symmetric $\tau \# \iota^*S_{\mathbb{Q}}\text{-mod}$ -algebra. ■

Chapter 9

Understanding $\tau \# \iota^* S_{\mathbb{Q}}\text{-mod}$

This chapter begins the work of proving that the methods of [GS] are compatible with the involution on $SO(2)$ -spectra. We are able to prove that there is a zig-zag of involutory Quillen equivalences between $\iota^* S_{\mathbb{Q}}\text{-mod}$ and a category $\text{mod-}\mathcal{E}_t$. Thus we have extended [GS, Theorem 4.1] (which is similar to Theorem 4.3.3) to the case of cyclic $O(2)$ -spectra. Furthermore, it should be possible to continue this work and extend the rest of [GS] to the case of cyclic spectra, see Remark 9.4.5. We begin by proving that $\text{mod-}\mathcal{E}_{top}$ is a category with involution. In Section 9.2 we prove that the Morita equivalence of Theorem 5.4.3 is involutory in the case of $SO(2)$ -spectra. In Section 9.3 we then prove that the functors of [Shi07b] are compatible with the involutions. The last section is another Morita equivalence, which is involutory by the work of the second section. The conclusion of this chapter is Corollary 9.4.4.

9.1 An Involution on $\text{mod-}\mathcal{E}_{top}$

The method of [GS] begins by replacing rational $SO(2)$ -spectra by $\text{mod-}\mathcal{E}_{top}$, which may be called the category of topological $SO(2)$ -Mackey functors. We show that this category has an involution in Proposition 9.1.8.

Consider the homogenous spaces $\Sigma^{\infty} SO(2)/H_+$ where H runs over all subgroups of $SO(2)$, these are a set of generators for $SO(2)\mathcal{M}$. Now we smash these with the rational $SO(2)$ -equivariant sphere spectrum, $\iota^* S_{\mathbb{Q}}$, to obtain a set of generators for $\iota^* S_{\mathbb{Q}}\text{-mod}$. We can now apply the idempotents and take cofibrant replacements (in the category of $\iota^* S_{\mathbb{Q}}$ -modules) to obtain the basic cells $\sigma_H = \widehat{ce}_H(SO(2)/H_+) \wedge \iota^* S_{\mathbb{Q}}$, since all spectra are fibrant in this model category, we have a set of cofibrant-fibrant objects which we call \mathcal{BC} . The following result implies that this collection is a generating set.

Lemma 9.1.1 *The homogenous spaces can be obtained from the basic cells.*

$$\Sigma^{\infty} SO(2)/H_+ \wedge \iota^* S_{\mathbb{Q}} \simeq \bigvee_{K \subseteq H} \sigma_K$$

Proof See [Gre99, Lemma 2.1.5]. ■

We have an alternative construction of the basic cells, which will be of use later. We will perform some of this work in the categories of rational D_{2n} -spectra and rational C_n -spectra. By similar arguments to the $SO(2)$ case we can use the inversion map of C_n to create an involution on rational C_n -spectra. We can then construct a model category of cyclic D_{2n} -spectra and see that this is Quillen equivalent to the skewed category $\tau \# C_n \mathcal{M}_{\mathbb{Q}}$. There is an idempotent $e_{C_n}^{D_{2n}} \in [S, S]_{\mathbb{Q}}^{D_{2n}}$ corresponding to the subgroup C_n of D_{2n} . This gives an idempotent in $[S \wedge EW_+, S \wedge EW_+]_{\mathbb{Q}}^{D_{2n}}$. This group is isomorphic to self maps of S in the homotopy category of cyclic D_{2n} -spectra. Thus, we have an idempotent $e'_{C_n} \in \text{Ho}(\tau \# C_n \mathcal{M}_{\mathbb{Q}})(S, S)$, maps in the homotopy category of the skewed category of C_n -spectra from the unit to itself.

We can choose a map in the skewed category of C_n -spectra representing e'_{C_n} . By applying $SO(2) \wedge_{C_n} (-)$ we obtain an idempotent map (up to homotopy) of $SO(2)/C_n \rightarrow h^{\dagger} j^* SO(2)/C_n$ in $\tau \# SO(2) \mathcal{M}$. We take homotopy colimits to construct the skewed object $e_{C_n} SO(2)/C_n \rightarrow h^{\dagger} j^* e_{C_n} SO(2)/C_n$. By smashing with $(\iota^* S_{\mathbb{Q}}, \alpha)$ and taking a cofibrant replacement we obtain a basic cell $w_H: \sigma_H \rightarrow \tau \sigma_H$ in $\tau \# \iota^* S_{\mathbb{Q}}\text{-mod}$. Since a cofibrant object of the skewed category is cofibrant in the underlying category, σ_H is a construction of a basic cell for $\iota^* S_{\mathbb{Q}}\text{-mod}$.

Now define $\bar{\mathcal{BC}}$ to be the closure of \mathcal{BC} under smash products with the unit included. By monoidality all non-unit objects are cofibrant and all objects are fibrant. Recall that we are using the smash product of $\iota^* S_{\mathbb{Q}}$ -modules for this definition. The full subcategory of $\iota^* S_{\mathbb{Q}}\text{-mod}$ with object set $\mathcal{G}_{top} := \bar{\mathcal{BC}}$ will be denoted \mathcal{E}_{top} .

Theorem 9.1.2 *The Quillen pair*

$$(-) \wedge_{\mathcal{E}_{top}} \mathcal{G}_{top} : \iota^* S_{\mathbb{Q}}\text{-mod} \rightleftarrows \text{mod-}\mathcal{E}_{top} : \underline{\text{Hom}}(\mathcal{G}_{top}, -)$$

is a strong symmetric monoidal Quillen equivalence.

Proof This is part of [GS, Theorem 4.1], we give some details of the proof. The result is essentially an application of Theorem 5.4.3 but we must adjust the proof slightly since now the unit $\iota^* S_{\mathbb{Q}} \in \mathcal{G}_{top}$ is not cofibrant. The functor $\underline{\text{Hom}}_{\iota^* S_{\mathbb{Q}}}(\iota^* S_{\mathbb{Q}}, -)$ preserves fibrations and all weak equivalences (since every object of $\iota^* S_{\mathbb{Q}}\text{-mod}$ is fibrant), hence the above adjunction is a Quillen pair. It is a Quillen equivalence by the same arguments of [SS03b, Theorem 3.9.3], with the following alterations.

The free modules F_{σ} are no longer cofibrant, however, as mentioned in Theorem 5.3.9 $\widehat{c} \iota^* S_{\mathbb{Q}} \wedge_{\iota^* S_{\mathbb{Q}}} F_{\sigma}$ is a cofibrant replacement. The left derived functor, $(-) \wedge_{\mathcal{E}_{top}}^L \mathcal{G}$, takes F_{σ} to $\widehat{c} \iota^* S_{\mathbb{Q}} \wedge_{\iota^* S_{\mathbb{Q}}} \sigma$. Since σ is either $\iota^* S_{\mathbb{Q}}$ or cofibrant, this is weakly equivalent to σ . We also note that since $\iota^* S_{\mathbb{Q}}\text{-mod}$ is a monoidal model category, the map $\widehat{c} \iota^* S_{\mathbb{Q}} \wedge_{\iota^* S_{\mathbb{Q}}} M \rightarrow \iota^* S_{\mathbb{Q}} \wedge_{\iota^* S_{\mathbb{Q}}} M$ is a weak equivalence for any cofibrant module M . Thus, by [Hov99, Lemma 4.2.7] the map $\underline{\text{Hom}}_{\iota^* S_{\mathbb{Q}}}(\iota^* S_{\mathbb{Q}}, M) \rightarrow \underline{\text{Hom}}_{\iota^* S_{\mathbb{Q}}}(\widehat{c} \iota^* S_{\mathbb{Q}}, M)$ is a weak equivalence for all $\iota^* S_{\mathbb{Q}}$ -modules M . Hence $\underline{\text{Hom}}_{\iota^* S_{\mathbb{Q}}}(\iota^* S_{\mathbb{Q}}, M)$ has the correct homotopy type. ■

Lemma 9.1.3 *The functor $\tau = \alpha^* h^{\dagger} j^*$ is a spectral functor, moreover (τ, τ) is an adjunction of closed symmetric monoidal spectral functors.*

Proof We prove that τ is an adjunction of closed symmetric monoidal Sp_+^{Σ} -algebras. Let K be a symmetric spectrum, then $K \mapsto \iota^* S_{\mathbb{Q}} \wedge \iota^* \mathbb{N}i_* \varepsilon_{O(2)}^* \mathbb{P}|K|$ defines a symmetric monoidal Quillen functor from Sp_+^{Σ} to $\iota^* S_{\mathbb{Q}}\text{-mod}$. Furthermore, there is a natural isomorphism of order two $\iota^* S_{\mathbb{Q}} \wedge \iota^* \mathbb{N}i_* \varepsilon_{O(2)}^* \mathbb{P}|K| \rightarrow \alpha^* h^{\dagger} j^* (\iota^* S_{\mathbb{Q}} \wedge \iota^* \mathbb{N}i_* \varepsilon_{O(2)}^* \mathbb{P}|K|)$, this comes from the map α on $\iota^* S_{\mathbb{Q}}$ and the $O(2)$ structure of $\mathbb{N}i_* \varepsilon_{O(2)}^* \mathbb{P}|K|$. ■

Corollary 9.1.4 *The functor τ is a self-inverse map of ringoid spectra $\tau: \mathcal{E}_{top} \rightarrow \tau \mathcal{E}_{top}$.*

Proof We define the set $\tau \mathcal{E}_{top}$ to be the full subcategory of $\iota^* S_{\mathbb{Q}}\text{-mod}$ with object set $\tau \mathcal{G}_{top}$. The result is then obvious and we do not introduce any new notation for the inverse functor $\tau: \tau \mathcal{E}_{top} \rightarrow \mathcal{E}_{top}$. ■

Lemma 9.1.5 *There is an invertible map of ringoid spectra $W: \mathcal{E}_{top} \rightarrow \tau \mathcal{E}_{top}$.*

Proof This is where we use our new construction of the basic cells which come with maps of order two: $w_H: \sigma_H \rightarrow \tau \sigma_H$. On objects, W acts as τ , so $W\sigma = \tau\sigma$. On the homomorphism spectra W acts as

$$\underline{\text{Hom}}_{Sp^{\Sigma}}(\tau w, w'): \underline{\text{Hom}}_{Sp^{\Sigma}}(\sigma, \sigma') \rightarrow \underline{\text{Hom}}_{Sp^{\Sigma}}(\tau \sigma, \tau \sigma')$$

(recall that σ is some smash product of the basic cells and we have defined w_H for each basic cell σ_H). It should be obvious that this defines a ringoid map. We denote inverse of this map as W^{-1} and this acts τ on objects and acts as on homomorphism spectra as $\underline{\text{Hom}}_{Sp^{\Sigma}}(w, \tau w')$. ■

To simplify our notation we now write $\underline{\text{Hom}}(X, Y)$ in the place of $\underline{\text{Hom}}_{Sp^{\Sigma}}(X, Y)$, for X and Y in $\iota^* S_{\mathbb{Q}}\text{-mod}$. We also write $\mathcal{E}_{top}(a, b)$ for $\underline{\text{Hom}}_{Sp^{\Sigma}}(a, b)$ when $a, b \in \mathcal{G}_{top}$

Lemma 9.1.6 *The functors τ and W as defined above are morphisms of symmetric monoidal Sp_+^{Σ} -categories.*

Proof Because τ is a strong monoidal functor and by the definition of W it is obvious that these maps are compatible with the monoidal structure on \mathcal{E}_{top} . Hence we have a commuting diagram for $F = \tau$ and $F = W$.

$$\begin{array}{ccc} \mathcal{E}_{top}(\sigma_2, \sigma_3) \wedge \mathcal{E}_{top}(\sigma_1, \sigma_2) & \longrightarrow & \mathcal{E}_{top}(\sigma_1, \sigma_3) \\ \downarrow F & & \downarrow F \\ (\tau \mathcal{E}_{top})(\tau \sigma_2, \tau \sigma_3) \wedge (\tau \mathcal{E}_{top})(\tau \sigma_1, \tau \sigma_2) & \longrightarrow & (\tau \mathcal{E}_{top})(\tau \sigma_1, \tau \sigma_3) \end{array}$$

■

Definition 9.1.7 We define an involution on $\text{mod-}\mathcal{E}_{top}$ by $\rho = (\tau W)^*$. That this functor is self-inverse follows immediately from the relation

$$\tau \circ \mathcal{E}_{top}(\tau w, w') = \mathcal{E}_{top}(w, \tau w') \circ \tau.$$

Proposition 9.1.8 *The pair $(\text{mod-}\mathcal{E}_{top}, \rho)$ give a monoidal model category with involution that satisfies the monoid axiom.*

Proof Proposition 5.3.10 and Lemma 9.1.6 shows that ρ is a strong monoidal functor. We must prove that ρ is involutory monoidal and then the rest follows via the machinery of involutory categories. We must prove that for \mathcal{E}_{top} -modules M and N , the map $\rho M \square \rho N \rightarrow \rho(M \square N)$ is a map of order two. We draw $(\rho M \square \rho N)(x)$ as the coequaliser of the diagram below. The left hand vertical map is induced by the \mathcal{E}_{top} -action map of M and N and the right hand map by the monoidal product and composition of \mathcal{E}_{top} .

$$\begin{array}{c} \bigvee_{a,b,c,d \in \mathcal{G}_{top}} \rho M(a) \wedge \rho N(b) \wedge \mathcal{E}_{top}(a, c) \wedge \mathcal{E}_{top}(b, d) \wedge \mathcal{E}_{top}(x, c \wedge d) \\ \downarrow \downarrow \\ \bigvee_{e,f \in \mathcal{G}_{top}} \rho M(e) \wedge \rho N(f) \wedge \mathcal{E}_{top}(x, e \wedge f) \end{array}$$

Note that $\rho M(e) = M(e)$ as symmetric spectra, the ρ is to indicate that the \mathcal{E}_{top} -action is different. Our map from this expression to $\rho(M \square N)$ is induced by the maps $\text{Id}_M \wedge \text{Id}_N \wedge \tau W_{(a,c)} \wedge \tau W_{(b,d)} \wedge \tau W_{(x,c \wedge d)}$ and $\text{Id}_M \wedge \text{Id}_N \wedge \tau W_{(x,e \wedge f)}$. It is clear from the description that our involution is monoidal. \blacksquare

9.2 An Involutory Morita Equivalence

In Proposition 9.2.3 we prove that the Morita equivalence is involutory, so that the category of cyclic $O(2)$ -spectra is Quillen equivalent to the skewed category of $\text{mod-}\mathcal{E}_{top}$.

Lemma 9.2.1 *The following square commutes up to a natural isomorphism, β . The pair $(\underline{\text{Hom}}(\mathcal{G}_{top}, -), \beta)$ define an involutory functor.*

$$\begin{array}{ccc} \iota^* S_{\mathbb{Q}}\text{-mod} & \xrightarrow{\underline{\text{Hom}}(\mathcal{G}_{top}, -)} & \text{mod-}\mathcal{E}_{top} \\ \downarrow \tau & & \downarrow \rho \\ \iota^* S_{\mathbb{Q}}\text{-mod} & \xrightarrow{\underline{\text{Hom}}(\mathcal{G}_{top}, -)} & \text{mod-}\mathcal{E}_{top} \end{array}$$

Proof We consider an $\iota^* S_{\mathbb{Q}}$ -module X , moving along the top this gives the module $\rho \underline{\text{Hom}}(-, X)$, the bottom route produces $\underline{\text{Hom}}(-, \tau X)$. We define a natural isomorphism $\beta': \rho \underline{\text{Hom}}(-, \tau X) \rightarrow \underline{\text{Hom}}(-, X)$ by $\underline{\text{Hom}}(w, \text{Id}_X) \circ \tau$. Naturality of β' is clear and since the diagram below obviously commutes, β' is a map of \mathcal{E}_{top} -modules. The top horizontal composition is the action of \mathcal{E}_{top} on $\rho \underline{\text{Hom}}(-, \tau X)$ and the bottom is the action of \mathcal{E}_{top} on $\underline{\text{Hom}}(-, X)$. From β' we have a natural transformation $\beta: \underline{\text{Hom}}(-, \tau X) \rightarrow \rho \underline{\text{Hom}}(-, X)$. Since applying β' twice gives the identity map $(\underline{\text{Hom}}(\mathcal{G}_{top}, -), \beta)$ is an involutory functor.

$$\begin{array}{ccccc}
 \underline{\text{Hom}}(A, \tau X) \wedge \mathcal{E}_{top}(B, A) & \xrightarrow{\text{Id} \wedge \tau \circ W} & \underline{\text{Hom}}(A, \tau X) \wedge \mathcal{E}_{top}(B, A) & \longrightarrow & \underline{\text{Hom}}(B, \tau X) \\
 \downarrow \tau \wedge \text{Id} & & \downarrow \tau \wedge \tau & & \downarrow \tau \\
 \underline{\text{Hom}}(\tau A, X) \wedge \mathcal{E}_{top}(B, A) & \xrightarrow{\text{Id} \wedge W} & \underline{\text{Hom}}(\tau A, X) \wedge (\tau \mathcal{E}_{top})(\tau B, \tau A) & \longrightarrow & \underline{\text{Hom}}(\tau B, X) \\
 \downarrow w_A^* \wedge \text{Id} & & & & \downarrow w_B^* \\
 \underline{\text{Hom}}(A, X) \wedge \mathcal{E}_{top}(B, A) & \longrightarrow & & \longrightarrow & \underline{\text{Hom}}(B, X)
 \end{array}$$

■

Now it follows by Lemma 7.1.17 that the left adjoint to $\underline{\text{Hom}}(\mathcal{G}_{top}, -)$ is an involutory functor. We construct a natural transformation α to prove this directly, since this type of construction will occur again. Pick M , an object of $\text{mod-}\mathcal{E}_{top}$, then $M \wedge_{\mathcal{E}_{top}} \mathcal{G}_{top}$ is given by the coequaliser in Definition 5.4.2. The natural transformation

$$\alpha: \rho(-) \wedge_{\mathcal{E}_{top}} \mathcal{G}_{top} \longrightarrow \tau((-) \wedge_{\mathcal{E}_{top}} \mathcal{G}_{top})$$

is defined by the map of coequalisers given in the diagram below, where μ is the action of \mathcal{E}_{top} on M .

$$\begin{array}{ccc}
 \bigvee_{g,h \in \mathcal{G}} M(h) \wedge \mathcal{E}_{top}(g, h) \wedge g & \xrightarrow[\text{eval}]{\mu \circ \tau \circ W_{g,h}} & \bigvee_{g \in \mathcal{G}} M(g) \wedge g \\
 \text{Id} \wedge \tau \circ W_{g,h} \wedge w_g \downarrow & & \downarrow \text{Id} \wedge w_g \\
 \bigvee_{g,h \in \mathcal{G}} M(h) \wedge \mathcal{E}_{top}(g, h) \wedge \tau g & \xrightarrow[\tau(\text{eval}) \circ \gamma]{\mu} & \bigvee_{g \in \mathcal{G}} M(g) \wedge \tau g
 \end{array}$$

To prove that the above does define a map of coequalisers it suffices to show that the two diagrams below commute.

$$\begin{array}{ccc}
 M(h) \wedge \mathcal{E}_{top}(g, h) & \xrightarrow{\mu \circ \tau \circ W_{g,h}} & M(g) \\
 \text{Id} \wedge \tau \circ W_{g,h} \downarrow & & \downarrow \text{Id} \\
 M(h) \wedge \mathcal{E}_{top}(g, h) & \xrightarrow{\mu} & M(g)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{E}_{top}(g, h) \wedge g & \xrightarrow{\text{eval}} & h \\
 \tau \circ W_{g,h} \wedge w_g \downarrow & & \downarrow w_h \\
 \mathcal{E}_{top}(g, h) \wedge \tau g & \xrightarrow{\tau(\text{eval}) \circ \gamma} & \tau h
 \end{array}$$

The left hand diagram automatically commutes. To see that the right hand diagram commutes we need Lemma 9.2.2. Which we have included to show how the various maps in the diagram are related. Note that this natural transformation α clearly satisfies the necessary condition for it to be an involutory natural transformation.

Lemma 9.2.2 *The diagram below commutes for all $g \in \mathcal{G}_{top}$ and is natural in $\iota^* S_{\mathbb{Q}}$ -modules X .*

$$\begin{array}{ccccc}
 & & g \wedge \underline{\text{Hom}}(\tau g, X) & & \\
 & \text{Id} \wedge (\tau w_g)^* \nearrow & & \text{Id} \wedge \tau \searrow & \\
 g \wedge \underline{\text{Hom}}(g, X) & & & & g \wedge \underline{\text{Hom}}(g, \tau X) \\
 & \text{Id} \wedge \tau \searrow & (1) & \text{Id} \wedge (w_g)^* \nearrow & \\
 & & g \wedge \underline{\text{Hom}}(\tau g, \tau X) & & \\
 \tau w_g \wedge \text{Id} \uparrow & (2) & \tau w_g \wedge \text{Id} \uparrow & (3) & \text{eval} \downarrow \\
 & & \tau g \wedge \underline{\text{Hom}}(\tau g, \tau X) & & \\
 \text{Id} \wedge \tau \nearrow & & & \text{eval} \searrow & \\
 \tau g \wedge \underline{\text{Hom}}(g, X) & & (4) & & \tau X \\
 & \gamma \searrow & & \tau(\text{eval}) \nearrow & \\
 & & \tau(g \wedge \underline{\text{Hom}}(g, X)) & &
 \end{array}$$

Proof The map labelled γ is the natural transformation between $\tau X \otimes A \rightarrow \tau(X \otimes A)$ for a symmetric spectrum A and a rational $\iota^* S_{\mathbb{Q}}$ -module X . Squares (1) and (2) obviously commute, square (4) commutes since τ is a spectral functor (see [Bor94, Proposition 6.4.5]). The fact that Square (3) commutes requires a little more consideration but essentially follows from axiom that evaluation and composition are compatible. ■

Proposition 9.2.3 *The adjunction $((-) \wedge_{\mathcal{E}_{top}} \mathcal{G}_{top}, \underline{\text{Hom}}(\mathcal{G}_{top}, -))$ is involutory.*

Proof We must prove that the unit η and counit ε are involutory natural transformations. For the counit, let X be an $\iota^* S_{\mathbb{Q}}$ -module. We must prove that the following diagram commutes.

$$\begin{array}{ccc}
 \int^{a \in \mathcal{G}_{top}} \underline{\text{Hom}}(a, \tau X) \wedge a & \xrightarrow{\varepsilon_{\tau X}} & \tau X \\
 \int^{a \in \mathcal{G}_{top}} \beta_X(a) \wedge \text{Id}_a \downarrow & & \tau(\varepsilon_X) \uparrow \\
 \int^{a \in \mathcal{G}_{top}} \rho \underline{\text{Hom}}(a, X) \wedge a & \xrightarrow{\alpha_{\underline{\text{Hom}}(\mathcal{G}_{top}, X)}} & \tau \int^{a \in \mathcal{G}_{top}} \underline{\text{Hom}}(a, X) \wedge a
 \end{array}$$

This is routine to check using Lemma 9.2.2. Now we consider the unit, let M be an \mathcal{E}_{top} -module. We must prove that the diagram below commutes.

$$\begin{array}{ccc}
 \rho M & \xrightarrow{\eta_{\rho M}} & \underline{\text{Hom}}(\mathcal{G}_{top}, \int^g \rho M(g) \wedge g) \\
 \rho \eta_M \uparrow & & \downarrow \underline{\text{Hom}}(\mathcal{G}_{top}, \alpha_M) \\
 \rho \underline{\text{Hom}}(\mathcal{G}_{top}, \int^g M(g) \wedge g) & \xleftarrow{\beta_{\int^g M(g) \wedge g}} & \underline{\text{Hom}}(\mathcal{G}_{top}, \tau \int^g M(g) \wedge g)
 \end{array}$$

Checking that this diagram commutes reduces to proving that the diagram below commutes.

$$\begin{array}{ccc}
 M(k) & \xrightarrow{\text{coeval}} & \underline{\text{Hom}}(k, k \wedge M(k)) \\
 \text{coeval} \downarrow & & \downarrow (w_k \wedge \text{Id})_* \\
 \underline{\text{Hom}}(k, k \wedge M(k)) & \xrightarrow{\gamma^{-1} \circ (w_k)^* \circ \tau} & \underline{\text{Hom}}(k, \tau k \wedge M(k))
 \end{array}$$

This is the coevaluation version of Lemma 9.2.2 and it commutes since the pair of squares below commute.

$$\begin{array}{ccc}
 M(k) & \xrightarrow{\text{coeval}_k} & \underline{\text{Hom}}(k, k \wedge M(k)) \\
 \text{coeval}_{\tau k} \downarrow & & \downarrow (w_k \wedge \text{Id})_* \\
 \underline{\text{Hom}}(\tau k, \tau k \wedge M(k)) & \xrightarrow{(w_k)^*} & \underline{\text{Hom}}(k, \tau k \wedge M(k))
 \end{array}$$

$$\begin{array}{ccc}
 M(k) & \xrightarrow{\text{coeval}_k} & \underline{\text{Hom}}(k, k \wedge M(k)) \\
 \text{coeval}_{\tau k} \downarrow & & \downarrow \tau \\
 \underline{\text{Hom}}(\tau k, \tau k \wedge M(k)) & \xrightarrow{\gamma_*} & \underline{\text{Hom}}(\tau k, \tau(k \wedge M(k)))
 \end{array}$$

■

Corollary 9.2.4 *There is a strong symmetric monoidal Quillen equivalence between the skewed categories*

$$\tau \# (\iota^* S_{\mathbb{Q}}\text{-mod}) \rightleftarrows \rho \# (\text{mod-} \mathcal{E}_{\text{top}}).$$

Furthermore this is an adjunction of closed symmetric $\text{Id} \# Sp_+^{\Sigma}$ -algebras.

Proof The results of this section prove that the Quillen equivalence of Theorem 9.1.2 is an involutory Quillen equivalence. Furthermore this is a monoidal involutory adjunction, though we omit the routine proof that the natural transformation β is a monoidal natural transformation. It remains to prove that in the diagram below (which gives the algebra structure on the skewed categories) the right adjoints (shown on the bottom) commute up to natural isomorphism.

$$\begin{array}{ccc}
 \rho \# (\text{mod-} \mathcal{E}_{\text{top}}) & \rightleftarrows & \tau \# \iota^* S_{\mathbb{Q}}\text{-mod} \\
 \swarrow \text{Ev}_{\iota^* S_{\mathbb{Q}}} & & \nearrow i \\
 & \text{Id} \# Sp_+^{\Sigma} & \\
 \nwarrow F_{\iota^* S_{\mathbb{Q}}} & & \searrow j
 \end{array}$$

Since the corresponding statement is true for the underlying categories all we need check is that the involutions are compatible, in the sense that for $x \in Sp_+^{\Sigma}$ and $E \in \iota^* S_{\mathbb{Q}}\text{-mod}$

the following diagrams commute.

$$\begin{array}{ccc}
 \int^g(\underline{\text{Hom}}(g, \iota^*S_{\mathbb{Q}}) \wedge x) \wedge g & \longrightarrow & \iota^*S_{\mathbb{Q}} \wedge x \\
 \downarrow & & \downarrow \\
 \int^g \rho(\underline{\text{Hom}}(g, \iota^*S_{\mathbb{Q}}) \wedge x) \wedge g & & \rho \underline{\text{Hom}}(\iota^*S_{\mathbb{Q}}, \tau E) \longrightarrow j\tau E \\
 \downarrow & & \downarrow \\
 \tau \int^g(\underline{\text{Hom}}(g, \iota^*S_{\mathbb{Q}}) \wedge x) \wedge g & \longrightarrow & \tau(\iota^*S_{\mathbb{Q}} \wedge x)
 \end{array}$$

The left hand diagram commutes by the same arguments we used above to show that $(-)\wedge_{\mathcal{E}_{top}} \mathcal{G}_{top}$ is an involutory functor. The right hand diagram commutes because τ does not alter the underlying symmetric spectrum of an $\iota^*S_{\mathbb{Q}}$ -module. ■

9.3 Moving to $\text{mod-}\Theta''\mathcal{E}_{top}$

Having shown that the equivalence between rational $SO(2)$ -spectra and $\text{mod-}\mathcal{E}_{top}$ is involutory, we now must show that the zig-zag of equivalences between $\text{mod-}\mathcal{E}_{top}$ and $\text{mod-}\Theta''\mathcal{E}_{top}$ (as proved in [GS, Theorem 4.1]) is involutory. There are several steps in this section, but they are all essentially the same. We begin with the easiest case where we move from enrichments over positive symmetric spectra to symmetric spectra. The main result of this section is Theorem 9.3.4.

Lemma 9.3.1 *The Quillen equivalence between the category of \mathcal{E}_{top} -modules over the model category of positive symmetric spectra and the category of \mathcal{E}_{top} -modules over the model category of symmetric spectra is an involutory strong symmetric monoidal Quillen equivalence.*

Proof Recall that there is a Quillen equivalence $\text{Id} : Sp_+^{\Sigma} \rightleftarrows Sp^{\Sigma} : \text{Id}$. We perform two operations, first we consider \mathcal{E}_{top} as a category enriched over symmetric spectra and secondly we consider the category of Sp^{Σ} -functors from \mathcal{E}_{top}^{op} to Sp^{Σ} . Thus we have the model category of \mathcal{E}_{top} -modules over Sp_+^{Σ} . The involution on this category is as before, it is given by a map of the ringoid $\mathcal{E}_{top} \rightarrow \mathcal{E}_{top}$, so we have an involutory adjunction.

Since the only difference between this new model category and $\text{mod-}\mathcal{E}_{top}$ as before is in the model structures, there is little to check in order to prove the result. The right adjoint preserves fibrations and the weak equivalences are the same for either model structure so we have a Quillen equivalence. ■

Since the category of \mathcal{E}_{top} -modules over the model category of symmetric spectra is only used briefly, we do not introduce any new notation for it, but it is essential for the remaining results that we are now using this model structure on \mathcal{E}_{top} -modules. In particular, the unit is now cofibrant. We now give some of the material in [Shi07b] since we will need to examine this in some detail to prove that the equivalence between \mathcal{E}_{top} -modules and \mathcal{E}_t -modules is involutory. We give the proposition first and then explain the terms in it.

Proposition 9.3.2 *The following series of adjoint pairs are Quillen equivalences.*

$$\begin{aligned} Q : \mathbb{H}\mathbb{Q}\text{-mod} &\rightleftarrows Sp^{\Sigma}(s\mathbb{Q}\text{-mod}) : U_1 \\ L : Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+) &\rightleftarrows Sp^{\Sigma}(s\mathbb{Q}\text{-mod}) : \phi^*N \\ D : Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+) &\rightleftarrows dg\mathbb{Q}\text{-mod} : R. \end{aligned}$$

Furthermore, the pair (Q, U_1) is strong symmetric monoidal, (L, ϕ^*N) is symmetric monoidal and (D, R) is strong monoidal. Each of the right adjoints preserve all weak equivalences.

Proof This is [Shi07b, Proposition 2.10], where we note that D is not symmetric monoidal, as explained in [Shi07a]. \blacksquare

We begin with the functor $\tilde{\mathbb{Q}} : s\text{Set} \rightarrow s\mathbb{Q}\text{-mod}$. For a simplicial set X , we define $(\tilde{\mathbb{Q}}X)_n$ to be the free \mathbb{Q} -module on the non-basepoint simplices of X_n , with each $0.s$ identified with the basepoint for $s \in X_n$. We fix the object $\tilde{\mathbb{Q}}S^1$ and use this as the suspension object to create the category $Sp^{\Sigma}(s\mathbb{Q}\text{-mod})$ of symmetric spectra in simplicial \mathbb{Q} -modules. The functor $\tilde{\mathbb{Q}}$ induces an adjoint pair $\tilde{\mathbb{Q}} : Sp^{\Sigma} \rightleftarrows Sp^{\Sigma}(s\mathbb{Q}\text{-mod}) : U$. Let $\mathbb{H}\mathbb{Q}$ be the symmetric spectrum with level n given by $\tilde{\mathbb{Q}}S^n$. This is a commutative ring spectrum, hence we have the category of $\mathbb{H}\mathbb{Q}$ -modules in symmetric spectra, $\mathbb{H}\mathbb{Q}\text{-mod}$. There is a forgetful functor $U_1 : Sp^{\Sigma}(s\mathbb{Q}\text{-mod}) \rightarrow \mathbb{H}\mathbb{Q}\text{-mod}$. This has a left adjoint Q , but this is not needed for the work below since $\mathcal{E}_{top}\text{-mod}$ is enriched over Sp^{Σ} .

The category of symmetric spectra in non-negatively graded chain complexes, written as $Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$, has suspension object $\mathbb{Q}[1]$ (one copy of \mathbb{Q} in degree 1). The normalisation functor $N : s\mathbb{Q}\text{-mod} \rightarrow dg\mathbb{Q}\text{-mod}_+$ induces a functor $\phi^*N : Sp^{\Sigma}(s\mathbb{Q}\text{-mod}) \rightarrow Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$, with left adjoint L . The functor R takes a chain complex Y to the symmetric spectrum with $RY_n = C_0(Y \otimes \mathbb{Q}[m])$ and has a left adjoint D .

In the following result we will use the pair $\tilde{\mathbb{Q}} : Sp^{\Sigma} \rightleftarrows Sp^{\Sigma}(s\mathbb{Q}\text{-mod}) : U$ which are a Quillen pair, but not a Quillen equivalence. This result is a part of [GS, Theorem 4.1].

Proposition 9.3.3 *For each of the adjoint pairs $(\tilde{\mathbb{Q}}, U)$, (L, ϕ^*N) and (D, R) , the induced adjunction below is a Quillen equivalence.*

$$\begin{aligned} \tilde{\mathbb{Q}} : \text{mod-}\mathcal{E}_{top} &\rightleftarrows \text{mod-}\tilde{\mathbb{Q}}\mathcal{E}_{top} : U' \\ L' : \text{mod-}\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top} &\rightleftarrows \text{mod-}\tilde{\mathbb{Q}}\mathcal{E}_{top} : \phi^*N \\ D : \text{mod-}\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top} &\rightleftarrows \text{mod-}D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top} : R' \end{aligned}$$

Proof We use the construction of Proposition 5.1.8 (which is a simplification of [DS07, Proposition A.3b]) to obtain the categories $\tilde{\mathbb{Q}}\mathcal{E}_{top}$, $\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ and $D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ which are enriched over symmetric spectra in simplicial \mathbb{Q} -modules, symmetric spectra in positive chain complexes of \mathbb{Q} -modules and chain complexes of \mathbb{Q} -modules respectively. The induced adjunctions are defined in [SS03a, Section 3] and we give brief details below. Since $\tilde{\mathbb{Q}}$ and D are strong monoidal these pass to the categories of modules as above without change. The right adjoint ϕ^*N also passes directly to the module categories whereas all the other functors must be replaced. The right adjoints U' and

R' are defined via the unit map, we demonstrate for U' . Take a $\tilde{\mathbb{Q}}\mathcal{E}_{top}$ -module M , we must then give maps

$$\mathcal{E}_{top}(\sigma', \sigma) \wedge U'M(\sigma) \rightarrow U'M(\sigma').$$

We do so by applying the unit map $\mathcal{E}_{top}(\sigma', \sigma) \rightarrow U'\tilde{\mathbb{Q}}\mathcal{E}_{top}(\sigma', \sigma)$ and then using the monoidality of U' and the action map of M . We will define L' in the proof of Theorem 9.3.4.

The pair $(\tilde{\mathbb{Q}}, U')$ induce a Quillen pair between $\text{mod-}\mathcal{E}_{top}$ and $\text{mod-}\tilde{\mathbb{Q}}\mathcal{E}_{top}$. The free modules are a set of generators for these categories and these free modules have rational homotopy groups. It follows that the unit and counit for the derived adjunctions are equivalences on these generators, hence $(\tilde{\mathbb{Q}}, U')$ is a Quillen equivalence. The other two pairs are Quillen equivalences by [SS03a, Theorem 6.5]. ■

Theorem 9.3.4 *For each of the adjoint pairs $(\tilde{\mathbb{Q}}, U)$, (L, ϕ^*N) and (D, R) the induced adjunction $(\tilde{\mathbb{Q}}, U')$, (L', ϕ^*N) and (D, R') is an involutory Quillen equivalence on the categories of modules.*

Proof Because the construction of the ringoids $\tilde{\mathbb{Q}}\mathcal{E}_{top}$, $\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ and $D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ is functorial these ringoids come with self-inverse maps as follows.

$$\begin{array}{lll} \tilde{\mathbb{Q}}\tau W & : \tilde{\mathbb{Q}}\mathcal{E}_{top} & \longrightarrow \tilde{\mathbb{Q}}\mathcal{E}_{top} \\ \phi^*N\tilde{\mathbb{Q}}\tau W & : \phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top} & \longrightarrow \phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top} \\ D\phi^*N\tilde{\mathbb{Q}}\tau W & : D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top} & \longrightarrow D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top} \end{array}$$

We have the following diagram of adjoint pairs for the pair $(\tilde{\mathbb{Q}}, U')$.

$$\begin{array}{ccc} \text{mod-}\mathcal{E}_{top} & \xrightleftharpoons[U']{\tilde{\mathbb{Q}}} & \text{mod-}\tilde{\mathbb{Q}}\mathcal{E}_{top} \\ \tau W^* \updownarrow \tau W^* & & \tilde{\mathbb{Q}}\tau W^* \updownarrow \tilde{\mathbb{Q}}\tau W^* \\ \text{mod-}\mathcal{E}_{top} & \xrightleftharpoons[U']{\tilde{\mathbb{Q}}} & \text{mod-}\tilde{\mathbb{Q}}\mathcal{E}_{top} \end{array}$$

The vertical adjunctions are restriction and extension of scalars in the case where the map of ringoids is an isomorphism. This diagram commutes in the sense that the square consisting of left adjoints (on top and left) commutes and the square of right adjoints commutes. We obtain similar diagrams for (L', ϕ^*N) and (D, R') , we must now show that these pairs are involutory. In the case of $(\tilde{\mathbb{Q}}, U')$ the above squares of left adjoints and right adjoints commute precisely, as we now show. Take an \mathcal{E}_{top} -module M , then we have the following commutative diagram

$$\begin{array}{ccccc} & & \tilde{\mathbb{Q}}M(a) \wedge \tilde{\mathbb{Q}}\mathcal{E}_{top}(b, a) & & \\ & \nearrow \text{Id} \wedge \tilde{\mathbb{Q}}\tau W & & \searrow \cong & \\ \tilde{\mathbb{Q}}M(a) \wedge \tilde{\mathbb{Q}}\mathcal{E}_{top}(b, a) & & & & \tilde{\mathbb{Q}}(M(a) \wedge \mathcal{E}_{top}(b, a)) \longrightarrow \tilde{\mathbb{Q}}M(b) \\ & \searrow \cong & & \nearrow \tilde{\mathbb{Q}}(\text{Id} \wedge \tau W) & \\ & & \tilde{\mathbb{Q}}(M(a) \wedge \mathcal{E}_{top}(b, a)) & & \end{array}$$

with the top path the action map for $(\tilde{Q}\tau W)^*\tilde{Q}M$ and the lower path the action map for $\tilde{Q}((\tau W)^*M)$. So we have shown that \tilde{Q} is an involutory functor since it strictly commutes with the involutions $(\tau W)^*$ and $(\tilde{Q}\tau W)^*$. Now we consider the module-level right adjoint U' . For a $\tilde{Q}\mathcal{E}_{top}$ module N , the following composition defines the action of \mathcal{E}_{top} on $U'N$, from this it is clear that U' also strictly commutes with the involutions.

$$U'N(a) \wedge \mathcal{E}_{top}(b, a) \xrightarrow{\text{Id} \wedge \eta} U'N(a) \wedge U'\tilde{Q}\mathcal{E}_{top}(b, a) \rightarrow U'(N(a) \wedge \tilde{Q}\mathcal{E}_{top}(b, a)) \rightarrow U'N(b)$$

It is then obvious that (\tilde{Q}, U') is an involutory adjunction that is also a Quillen equivalence. The case (D, R') is exactly the same, which leaves us with only the adjunction (L', ϕ^*N) to consider. The right adjoint ϕ^*N strictly commutes with the involutions via the same arguments as for \tilde{Q} and D . We investigate L' in some detail, take a $\phi^*N\tilde{Q}\mathcal{E}_{top}$ -module M , then $L'M(a)$ is defined as the coequaliser of the following diagram (we describe the maps below).

$$\bigvee_{b,c} L(M(b) \wedge \phi^*N\tilde{Q}\mathcal{E}_{top}(c, b)) \wedge \tilde{Q}\mathcal{E}_{top}(a, c) \rightrightarrows \bigvee_d LM(d) \wedge \tilde{Q}\mathcal{E}_{top}(a, d)$$

One map is induced by the action of $\phi^*N\tilde{Q}\mathcal{E}_{top}$ on M and the other is the composite of the op-monoidal structure on L , the counit of (L, ϕ^*N) and composition. We can induce a map of coequalisers

$$L'(\phi^*N\tilde{Q}\tau W)^*M \longrightarrow (\tilde{Q}\tau W)^*L'M$$

by acting as $L(\text{Id} \wedge \phi^*N\tilde{Q}\tau W) \wedge \tilde{Q}\tau W$ on the first factor and by $\text{Id} \wedge \tilde{Q}\tau W$ on the second. This is clearly a morphism of modules and is a map order two as required. It remains to check that the unit and counit are involutory natural transformations, these maps are induced from the unit and counit of (L, ϕ^*N) and it is easy to check the required condition. The proof is very much like that for unit and counit of the Morita equivalence and we omit it for that reason. \blacksquare

Lemma 9.3.5 *The categories $\tilde{Q}\mathcal{E}_{top}$ and $\phi^*N\tilde{Q}\mathcal{E}_{top}$ are monoidal enriched categories. Furthermore, the adjunctions (\tilde{Q}, U') and (L', ϕ^*N) are involutory symmetric monoidal Quillen equivalences and hence induce symmetric monoidal Quillen equivalences on the skewed categories.*

Proof The first statement is an application of Proposition 5.1.8 since both \tilde{Q} and ϕ^*N are symmetric monoidal functors. We prove that (\tilde{Q}, U') and (L', ϕ^*N) are monoidal pairs on the involutory categories. It is easy to check that the two right adjoints U' and ϕ^*N are monoidal functors. Since \tilde{Q} is strong symmetric monoidal on the base categories it passes to a strong symmetric monoidal functor $\tilde{Q}: \text{mod-}\mathcal{E}_{top} \rightarrow \text{mod-}\tilde{Q}\mathcal{E}_{top}$. Thus (\tilde{Q}, U') is a strong symmetric monoidal pair.

The left adjoint L' is harder to deal with, we must show that $\eta: L'(\phi^*N\tilde{Q}\mathcal{E}_{top}(-, \mathbb{S})) \rightarrow \tilde{Q}\mathcal{E}_{top}(-, \mathbb{S})$ is a weak equivalence and that for cofibrant $\phi^*N\tilde{Q}\mathcal{E}_{top}$ -modules X and Y , the map $m: L'(X \square Y) \rightarrow L'X \square L'Y$ is a weak equivalence. For each $g \in \mathcal{E}_{top}$ there is a $\phi^*N\tilde{Q}\mathcal{E}_{top}$ -module $F_g = \phi^*N\tilde{Q}\mathcal{E}_{top}(-, g)$. By Remark 5.3.15 these give a Quillen pair $F_g \wedge (-): Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+) \rightleftarrows \text{mod-}\phi^*N\tilde{Q}\mathcal{E}_{top} : \text{Ev}_g$, where $\text{Ev}_g(M) = M(g)$ and the

left adjoint takes $X \in Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$ to $F_g \wedge X$. Since the unit of $Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$ is cofibrant, it follows that each F_g is cofibrant as a $\phi^* N \tilde{\mathbb{Q}}\mathcal{E}_{top}$ -module. The unit is also a free module, it is given by $F_{\iota^* S_{\mathbb{Q}}}$. For each $g \in \mathcal{E}_{top}$, there is a square as below. This will allow us to translate results from the base categories to the module categories and helps us understand the functor L' . The right adjoints are on the right and bottom and clearly commute.

$$\begin{array}{ccc}
 \text{mod-}\phi^* N \tilde{\mathbb{Q}}\mathcal{E}_{top} & \xrightleftharpoons[\phi^* N]{L'} & \text{mod-}\tilde{\mathbb{Q}}\mathcal{E}_{top} \\
 \updownarrow \begin{smallmatrix} F_g \wedge (-) \\ \text{Ev}_g \end{smallmatrix} & & \updownarrow \begin{smallmatrix} F_g \wedge (-) \\ \text{Ev}_g \end{smallmatrix} \\
 Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+) & \xrightleftharpoons[\phi^* N]{L} & Sp^{\Sigma}(s\mathbb{Q}\text{-mod})
 \end{array}$$

From this square we can check that η is a weak equivalence. Let $\text{Sym}(\mathbb{Q}[1])$ denote the unit of $Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$ and $\text{Sym}(\tilde{\mathbb{Q}}S^1)$ the unit of $Sp^{\Sigma}(s\mathbb{Q}\text{-mod})$, these are cofibrant objects. Then $L'F_{\iota^* S_{\mathbb{Q}}} \cong F_{\iota^* S_{\mathbb{Q}}} \wedge L\text{Sym}(\mathbb{Q}[1])$, so the map η is given by $F_{\iota^* S_{\mathbb{Q}}} \wedge L\text{Sym}(\mathbb{Q}[1]) \rightarrow F_{\iota^* S_{\mathbb{Q}}} \wedge \text{Sym}(\tilde{\mathbb{Q}}S^1)$. It follows that this map is an object-wise weak equivalence since cofibrant objects in $Sp^{\Sigma}(s\mathbb{Q}\text{-mod})$ preserve all weak equivalences (this is in the proof of [Shi07b, Corollary 3.4]).

We now give an argument to prove that we only need check that m is a weak equivalence on the free modules. We are required to prove that for all cofibrant X and Y , the map $m: L'(X \square Y) \rightarrow L'X \square L'Y$ is a weak equivalence. This is equivalent to proving that for all fibrant Z the map $[L'X \square L'Y, Z] \rightarrow [L'(X \square Y), Z]$ is a weak equivalence. By the standard adjunctions (such as the isomorphisms below) this occurs for all X exactly when $\phi^* N \text{Hom}_{\square}(L'Y, Z) \rightarrow \text{Hom}_{\square}(Y, \phi^* N Z)$ is a weak equivalence. To prove this, it suffices to show that the composite

$$[F_g, \phi^* N \text{Hom}_{\square}(L'Y, Z)] \cong [L'F_g \square L'Y, Z] \rightarrow [L'(F_g \square Y), Z] \cong [F_g, \text{Hom}_{\square}(Y, \phi^* N Z)]$$

is an isomorphism for the collection of free modules $F_g = \phi^* N \tilde{\mathbb{Q}}\mathcal{E}_{top}(-, g)$ as g runs over the set of objects in \mathcal{E}_{top} . Thus, we have shown that we only need prove that $L'(F_g \square Y) \rightarrow L'F_g \square L'Y$ is a weak equivalence for all g and all cofibrant Y . Applying the above argument once more we see that it suffices to prove that $L'(F_g \square F_k) \rightarrow L'F_g \square L'F_k$ is a weak equivalence for each g and k in \mathcal{E}_{top} . We do so now. Using our understanding of L' on free modules and the isomorphism $F_g \square F_k \rightarrow F_{(g \wedge k)}$ we must show that

$$F_{(g \wedge k)} \wedge L\text{Sym}(\mathbb{Q}[1]) \rightarrow F_{(g \wedge k)} \wedge L\text{Sym}(\mathbb{Q}[1]) \wedge L\text{Sym}(\mathbb{Q}[1])$$

is a weak equivalence. This follows from the corresponding result in $Sp^{\Sigma}(s\mathbb{Q}\text{-mod})$. One must also check that the natural transformations giving an involutory structure on $\phi^* N$ and U' are monoidal. This is quite straightforward due to the nature of the involutions. \blacksquare

Remark 9.3.6 Because D is not symmetric, as is stated in [Shi07a], the $dg\mathbb{Q}\text{-mod-}$ category $D\phi^* N \tilde{\mathbb{Q}}\mathcal{E}_{top}$ is not a monoidal $dg\mathbb{Q}\text{-mod-}$ category. In turn, the category

$\text{mod-}\mathcal{E}_t$ cannot be monoidal. This issue could be resolved by using the four stage comparison of [Shi07b, Remark 2.11] and altering the fibrant replacement functor. In detail, we replace the pair (D, R) by the functors

$$dg\mathbb{Q}\text{-mod} \xrightleftharpoons[\text{Ev}_0]{F_0} Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}) \xrightleftharpoons[C_0]{i} Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$$

The pair (F_0, Ev_0) are the suspension and zeroth space adjunction. The inclusion of positive chain complexes $i: dg\mathbb{Q}\text{-mod}_+ \rightarrow dg\mathbb{Q}\text{-mod}$ has a right adjoint C_0 . These are strong symmetric monoidal Quillen equivalences and i preserves all weak equivalences. We can then use these functors to create a symmetric monoidal enriched category $\text{Ev}_0 i\phi^* N\tilde{\mathcal{Q}}\mathcal{E}_{top}$. Unfortunately, since Ev_0 doesn't preserve all weak equivalences this will not have the correct homotopy type. One gets round this by inserting a fibrant replacement functor of $Sp^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$ -enriched categories as given by [SS03a, Proposition 6.3]. Thus $\text{mod-}\mathcal{E}_{top}$ is Quillen equivalent to $\text{mod-}\text{Ev}_0 \hat{f}i\phi^* N\tilde{\mathcal{Q}}\mathcal{E}_{top}$. But this is not a monoidal category, as \hat{f} will not preserve the monoidal product. It should be possible to alter this fibrant replacement so that it does preserve symmetric monoidal structures on enriched categories and ensure that $\text{Ev}_0 \hat{f}i\phi^* N\tilde{\mathcal{Q}}\mathcal{E}_{top}$ has the correct homotopy type. All of the functors relating $\text{mod-}\mathcal{E}_{top}$ and $\text{mod-}\mathcal{E}_t$ would then be both monoidal and involutory. An extra step would then be necessary: an adjunction of extension and restriction of scalars induced by the quasi-isomorphism $i\phi^* N\tilde{\mathcal{Q}}\mathcal{E}_{top} \rightarrow \hat{f}i\phi^* N\tilde{\mathcal{Q}}\mathcal{E}_{top}$, but this would present no difficulty.

Lemma 9.3.7 *If $L : M \rightleftharpoons N : R$ is a Quillen module over $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ and (L, R) is an involutory adjunction, then $\sigma \# L : \sigma \# M \rightleftharpoons \tau \# N : \tau \# R$ is a Quillen module over $(\text{Id} \# F, \text{Id} \# G)$ provided the following holds.*

- (i). *There is a natural transformation of order two $(\sigma m) \otimes c \rightarrow \sigma(m \otimes c)$.*
- (ii). *There is a natural transformation of order two $(\tau n) \otimes d \rightarrow \tau(n \otimes d)$.*
- (iii). *The diagram below commutes.*

$$\begin{array}{ccc} L(\sigma m \otimes c) & \longrightarrow & L\sigma m \otimes Fc \\ \downarrow & & \downarrow \\ L\sigma(m \otimes c) & & \tau Lm \otimes Fc \\ \downarrow & & \downarrow \\ \tau L(m \otimes c) & \longrightarrow & \tau(Lm \otimes Fc) \end{array}$$

Proof The first two conditions ensure that σ and τ are enriched functors, so that $\sigma \# M$ and $\tau \# N$ are modules over $\text{Id} \# \mathcal{C}$ and $\text{Id} \# \mathcal{D}$ respectively. One must check the conditions of [DS07, Propositions 3.5, 3.6, 3.7] to see that we have a Quillen adjoint module of the skewed categories. These conditions all hold due to their counterparts in M and N provided that the natural transformation $L(m \otimes c) \rightarrow Lm \otimes Fc$ induces a map on the skewed category, which is the third condition. Thus $(\sigma \# L, \tau \# R)$ is a Quillen adjoint module over $(\text{Id} \# F, \text{Id} \# G)$. \blacksquare

Corollary 9.3.8 *Each of the adjunctions of the skewed module categories of Proposition 9.3.3 is a Quillen module over the skewed base categories.*

Proof That the functors of Proposition 9.3.3 are Quillen modules at the level of involutory categories follows from [GS, Section 10]. We have worked from a more recent redraft of this paper, where this section has been altered to take into account [DS07, Propositions 4.7 and 4.8] which consider Quillen modules of categories of modules over enriched categories.

It remains to prove that the assumptions of Lemma 9.3.7 hold for the three adjunctions of Proposition 9.3.3. This is easy for $(\tilde{\mathbb{Q}}, U')$ and (D, R') since these adjoint pairs strictly commute with the involutions. The proof for $(L', \phi^* N)$ is routine. ■

9.4 Another Involutory Morita Equivalence

We show that one last Quillen equivalence is involutory: the Morita equivalence between $\text{mod-}D\phi^* N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ and $\text{mod-}\mathcal{E}_t$.

Definition 9.4.1 We define the category $SO(2)\text{-spectra}_t$ to be $\text{mod-}D\phi^* N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ and we will use λ for the involution on this category, hence we have the skewed category $\lambda\#SO(2)\text{-spectra}_t$. As in [GS, Theorem 4.1] we define \mathcal{BC}_t to be cofibrant replacements of the images of the basic cells of $SO(2)\text{-spectra}$ under the composite functor from $SO(2)\text{-spectra}$ to $SO(2)\text{-spectra}_t$. The closure of \mathcal{BC}_t under the monoidal product will be written $\tilde{\mathcal{BC}}_t$. Define \mathcal{E}_t to be the full subcategory of $SO(2)\text{-spectra}_t$ with object set $\tilde{\mathcal{BC}}_t$. The category \mathcal{E}_t is enriched over differential graded \mathbb{Q} -modules.

Thus the objects of \mathcal{BC}_t have the form $(\sigma_H)_t = \hat{c}D\phi^* N\tilde{\mathbb{Q}}Sp^{\Sigma}\underline{\text{Hom}}(-, \sigma_H)$ for σ_H a basic cell, where \hat{c} is cofibrant replacement in the skewed category.

For each H there is a map of order two $u_H: (\sigma_H)_t \rightarrow \lambda(\sigma_H)_t$ which is induced by the map $\sigma_H \rightarrow \tau\sigma_H$ and the natural transformations of the involutory functors in the composite. As with \mathcal{E}_{top} we have a ringoid $\lambda\mathcal{E}_t$ and conjugation by the u_H gives a map of ringoids $U: \mathcal{E}_t \rightarrow \lambda\mathcal{E}_t$.

Lemma 9.4.2 *The category $\text{mod-}\mathcal{E}_t$ is a model category with involution $(\lambda U)^*$.*

Proof The follows from the construction of the involution $\rho = (\tau W)^*$ on $\text{mod-}\mathcal{E}_{top}$ in Section 9.1. ■

Theorem 9.4.3 *The functors of Theorem 5.4.3 induce an involutory Quillen equivalence between $\text{mod-}\mathcal{E}_t$ and $SO(2)\text{-spectra}_t$. Thus there is a Quillen equivalence between the skewed categories*

$$(\lambda U)^* \# (\text{mod-}\mathcal{E}_t) \rightleftarrows \lambda \# SO(2)\text{-spectra}_t$$

and this is an adjunction of $(\text{Id} \# dg\mathbb{Q}\text{-mod})$ -modules.

Proof This follows from the proof of Corollary 9.2.4. ■

Corollary 9.4.4 *There is a zig-zag of Quillen equivalences between $\tau\#i^*S_{\mathbb{Q}}\text{-mod}$ and $(\lambda U)^*\#(\text{mod-}\mathcal{E}_t)$.*

Remark 9.4.5 Now we have a zig-zag of involutory equivalences between cyclic $O(2)$ -spectra and $\text{mod-}\mathcal{E}_t$. We describe our plan for future work. The standard category $\mathcal{A}(SO(2))$ is ‘formed’ from copies of the graded ring $\mathbb{Q}[c] \cong H^*(BSO(2))$ with c of degree 2 see [Gre99]. The inversion map $j: SO(2) \rightarrow SO(2)$, $j(t) = t^{-1}$, induces a ring map $\mathbb{Q}[c] \rightarrow \mathbb{Q}[c]$ which sends c to $-c$. Thus, we hope to create an involution $\Upsilon: \mathcal{A}(SO(2)) \rightarrow \mathcal{A}(SO(2))$ based on this ring map. Following the proof of Corollary 9.2.4 we should then be able to prove that $\mathcal{A}(SO(2))$ is Quillen equivalent to $\text{mod-}\mathcal{E}_a$ and that this equivalence is involutory monoidal. From here we aim to take the equivalences between \mathcal{E}_t and \mathcal{E}_a of [GS] and show that these are involutory (and monoidal according to the outcome of Remark 9.3.6). This would complete the classification of cyclic $O(2)$ -spectra in terms of the skewed category $\Upsilon\#\mathcal{A}(SO(2))$.

Appendix A

List of Model Categories

We list the model categories used in this thesis. We have divided this list into four collections, those which are categories of spectra, those we have constructed as localisations of such categories, categories of modules over enriched categories and a miscellany of basic categories.

Categories of Spectra

Name	Symbol	Page
Symmetric spectra in based simplicial sets	Sp^Σ	9
Sp^Σ with the positive model structure	Sp_+^Σ	10
Symmetric spectra in \mathcal{T}_*	$Sp_+^\Sigma(\mathcal{T}_*)$	10, 103
Symmetric spectra in $dg\mathbb{Q}\text{-mod}_+$	$Sp^\Sigma(dg\mathbb{Q}\text{-mod}_+)$	10, 113
Symmetric spectra in $s\mathbb{Q}\text{-mod}$	$Sp^\Sigma(s\mathbb{Q}\text{-mod})$	10, 113
G -equivariant orthogonal spectra	$G\mathcal{I}\mathcal{S}$	10, 12
G -equivariant S -modules	$G\mathcal{M}$	11, 13
$G\mathcal{I}\mathcal{S}$ with the positive model structure	$G\mathcal{I}\mathcal{S}_+$	13
Modules over $S_\mathbb{Q}$ in $O(2)\mathcal{M}$	$S_\mathbb{Q}\text{-mod}$	26, 35
Modules over S_H in $O(2)\mathcal{M}$	$S_H\text{-mod}$	55
Modules over $\iota^*S_\mathbb{Q}$ in $SO(2)\mathcal{M}$	$\iota^*S_\mathbb{Q}\text{-mod}$	99
The involutory category of $\iota^*S_\mathbb{Q}$	$(\iota^*S_\mathbb{Q}\text{-mod}, \tau)$	102
Modules over $H\mathbb{Q}$ in symmetric spectra	$H\mathbb{Q}\text{-mod}$	113

Categories of Localised Spectra

Name	Symbol	Page
$G\mathcal{I}\mathcal{S}$ localised at E , $L_E G\mathcal{I}\mathcal{S}$	$G\mathcal{I}\mathcal{S}_E$	28
GM localised at E , $L_E GM$	GM_E	28
$G\mathcal{I}\mathcal{S}$ localised at $S^0\mathbb{Q}$	$G\mathcal{I}\mathcal{S}_{\mathbb{Q}}$	32
$G\mathcal{I}\mathcal{S}_+$ localised at $S^0\mathbb{Q}_+$	$G\mathcal{I}\mathcal{S}_{\mathbb{Q}}^+$	32
GM localised at $S_{\mathcal{M}}^0\mathbb{Q}$	$GM_{\mathbb{Q}}$	32
$S_{\mathbb{Q}}\text{-mod}$ localised at $E \wedge S_{\mathbb{Q}}$	$L_E S_{\mathbb{Q}}\text{-mod}$	32
$L_{E\mathcal{F}_+} G\mathcal{I}\mathcal{S}$	$\mathcal{F}\mathcal{I}\mathcal{S}$	45
$L_{E\tilde{\mathcal{F}}} G\mathcal{I}\mathcal{S}$	$\tilde{\mathcal{F}}\mathcal{I}\mathcal{S}$	45
$L_{E\mathcal{F}_+} GM$	$\mathcal{F}\mathcal{M}$	45
$L_{E\tilde{\mathcal{F}}} GM$	$\tilde{\mathcal{F}}\mathcal{M}$	45
GM with $\mathcal{F}(N)$ -model structure	$GM(N)$	46
Cyclic $O(2)$ -spectra	$\mathcal{C}\mathcal{M}_{\mathbb{Q}}$	76
Dihedral $O(2)$ -spectra	$\mathcal{D}\mathcal{M}_{\mathbb{Q}}$	76
Cyclic $S_{\mathbb{Q}}$ -modules	$S_{\mathbb{Q}}\text{-mod}(\mathcal{C})$	78

Modules over an Enriched Category

Name	Symbol	Page
Right \mathcal{E}_a^H -modules in $dg\mathbb{Q}\text{-mod}$	$\text{mod-}\mathcal{E}_a^H$	52
Right \mathcal{E}_{top}^H -modules in Sp_+^{Σ}	$\text{mod-}\mathcal{E}_{top}^H$	55
Right modules over \mathcal{O} in Sp^{Σ}	$\text{mod-}\mathcal{O}$	68, 68
Right \mathcal{E}_{top} -modules in Sp_+^{Σ}	$\text{mod-}\mathcal{E}_{top}$	106
Right $\tilde{\mathbb{Q}}\mathcal{E}_{top}$ -modules in $Sp^{\Sigma}(\text{s}\mathbb{Q}\text{-mod})$	$\text{mod-}\tilde{\mathbb{Q}}\mathcal{E}_{top}$	113
Right $\phi^* N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ -modules in $Sp_+^{\Sigma}(dg\mathbb{Q}\text{-mod}_+)$	$\text{mod-}\phi^* N\tilde{\mathbb{Q}}\mathcal{E}_{top}$	113
Right $D\phi^* N\tilde{\mathbb{Q}}\mathcal{E}_{top}$ -modules in $dg\mathbb{Q}\text{-mod}$	$\text{mod-}D\phi^* N\tilde{\mathbb{Q}}\mathcal{E}_{top}$	113
Right \mathcal{E}_t -modules in $dg\mathbb{Q}\text{-mod}$	$\text{mod-}\mathcal{E}_t$	118

Miscellaneous

Name	Symbol	Page
Based topological spaces	\mathcal{T}_*	9
Based G -equivariant topological spaces	$G\mathcal{T}_*$	9
Based simplicial sets	sSet_*	9
Simplicial \mathbb{Q} -modules	$\text{s}\mathbb{Q}\text{-mod}$	9
Chain complexes of R -modules	$dgR\text{-mod}$	11
Positive chain complexes of R -modules	$dgR\text{-mod}_+$	12
Chain complexes of $\mathbb{Q}G$ -modules	$dg\mathbb{Q}G\text{-mod}$	51
A category with involution	(\mathcal{C}, σ)	79
The skewed category of (\mathcal{C}, σ)	$\sigma\#\mathcal{C}$	79

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- \mathcal{C} -module, 64
- \mathcal{C} -module functor, 64
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